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ADMIRALTY MANUAL OF TIDES

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ADMIRALTY MANUAL OF TIDES

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Published by the Hydrographic Department, Admiralty
To be obtained from the agents for the sale of Admiralty Charts

LONDON:
HER MAJESTY'S STATIONERY OFFICE
1941
(Reprinted 1961*)*
Price 15*s.* 0*d.* net

PREFACE

In recent years tidal theory has developed very rapidly and methods of analysis and predictions of tides have been improved to such a degree that the need has arisen for an exposition of these modern developments in a form suitable for instruction as well as general reading. This Manual has therefore been compiled with a view to the exposition of these matters, particularly for the benefit of hydrographic surveyors and naval officers in general. The adoption of the Admiralty method of predicting tides and tidal streams has made the need for such a book even more evident, and the authors of that method were therefore requested to prepare a general Manual, which would be intermediate in character between books too elementary for service needs and books which are too mathematical for officers with normal attainments in mathematics.

The Manual is divisible into four parts:

(1) The Theory of Tidal Forces and of Harmonic Methods (Chapters II to IX);

(2) Practical Problems of Recording, Analysing, and Predicting Tides (Chapters X to XVI);

(3) The Theory of Tidal Movements in Channels, Seas, and Oceans (Chapters XVII to XXV);

(4) The Theory of Special Tidal Phenomena (Chapters XXVI to XXIX);

The large subject of ocean currents has not been included, as it would involve many considerations pertaining more to Oceanography than to Tides, and having little bearing on tidal theory, but the more transient phenomena of meteorological surges has been considered in the Manual.

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to whom thanks are due.

J. A. EDGELL,
Vice Admiral and
Hydrographer of the Navy.

HYDROGRAPHIC DEPARTMENT,
ADMIRALTY, LONDON.
December, 1941.

FOREWORD BY THE AUTHORS

THE subject of tides cannot be adequately considered without mathematics, but, as this Manual has not been written for mathematicians and tidal specialists, the authors have endeavoured to confine the arguments to the limits imposed by a knowledge of elementary trigonometry. The results of abstruse mathematical investigations can be explained, within certain limits, by more elementary methods, and the authors decided that, where *a priori* proof by elementary methods is not possible, the results of the more advanced mathematical reasoning should be stated and examined to show their validity, and the mathematical investigations given in special articles marked with an asterisk (*). This plan has been followed throughout the book, so the general reader can evade all the mathematical investigations, but it must not be assumed that articles so marked are difficult to understand, and it is desirable that an effort should be made to follow the reasoning, especially in articles after Chapter XV.

The theory of tidal forces is developed by somewhat novel graphical methods (supplemented, of course, by the mathematical investigations) which reveal all the variations of forces and the characteristics of the tides. These methods are very simple yet very valuable for purposes of demonstration.

The development of the tide-generating forces necessarily requires some knowledge of the motions of the sun and moon, but the information is given in a simple manner, for the required formulæ are first quoted and their validity then tested by general reasoning. On the basis of these formulæ the principal harmonic constituents are deduced by simple methods and the mathematical development is reserved to a later chapter for the readers more interested in such technical matters. This development has been specially devised for this Manual with a view to the explanation and justification of the Admiralty method of analysis and prediction, which is critically examined in a later chapter in regard to its theoretical basis and also to its powers and limitations.

The discussion of the analysis and prediction of tides is confined to general principles, but special attention is given to the difficulties associated with the prediction of tides in shallow water.

The general theory dealing with the tidal movements in channels, seas and oceans has received very careful consideration, and special attention is called to original matter and methods not previously published. Lord Rayleigh's artifice of considering a progressive wave as a case of steady motion, by supposing the observer to travel with the wave, has been considerably extended so as to obtain quite simple expositions of many matters which have not previously been capable of exposition without the use of advanced mathematical methods. In particular, the method has been applied to (1) the distortion of progressive waves travelling in shallow water, leading up to the theory of shallow-water tides; (2) Kelvin waves; (3) "resonance" in long canals; (4) the theory of bores, which is novel and is here given for the first time; (5) meteorological surges; (6) internal waves. Some of these matters have not previously been expounded with any degree of satisfaction. Certain investigations of double tides are also original, and throw much light on an obscure subject which has given rise to many false theories.

The authors are greatly indebted to Professor Proudman for many leading ideas, some of which have not been published by him; for instance, the simple exposition of Ekman's theory of drift-currents is based upon his suggestions. They are also indebted to the staff of the Liverpool Observatory and Tidal Institute for much assistance.

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NOTE.—The chapters, articles, figures, tables and expressions are numbered as follows :—

CHAPTERS : in Roman numbers, thus " Chapter XIV," and when referred to elsewhere " see Chapter XIV."

ARTICLES, FIGURES and TABLES : by consecutive numbers in each chapter, preceded by the chapter number, both in Arabic numerals. When articles are referred to elsewhere the numbers are preceded by "Art."; the numbers are always preceded by " Fig." in the case of figures, and by " Table " in the case of tables.

Thus, the article numbered 14.3 is the third article in Chapter XIV, and is referred to outside the article as " Art. 14.3." Fig. 14.3 is the third figure in Chapter XIV and is not necessarily in Art. 14.3 ; Table 14.1 is the first table in Chapter XIV and is not necessarily in Art. 14.1.

EXPRESSIONS : by chapter and article numbers, as above, followed by a letter, the whole in brackets ; the letters are in alphabetical order in each article. Thus, (14.4g) is the seventh expression in the fourth article of Chapter XIV.

ADMIRALTY MANUAL OF TIDES

CHAPTER I

INTRODUCTION

1.1. Preliminary remarks

The flow and ebb of the tides have exercised human curiosity and reasoning powers for many centuries, and during the course of time many theories have been proposed, some of which have been creditable both to those who proposed them and to those who accepted them, though some have served only to show the credulity which so often afflicts men. There are few subjects which have been more associated with fantastic theories and speculations, and that not only in ancient times, nor only in the dark ages of a thousand years ago, but even in recent days. "It is not necessary," wrote Sir George Darwin in 1911, "to search ancient literatures for grotesque theories of the tides. In 1722, E. Barlow, gentleman, in "An exact survey of the Tide" attributes it to the pressure of the moon on the atmosphere. And theories, not less absurd, have been promulgated during the last 20 years."

Even in more recent times, it is not uncommon for tidal authorities to receive written or printed expositions of tidal theories which seek to deny the accumulated knowledge of the centuries on the ground that the writers are unable to understand it, or that it does not give perfect representation of the phenomena. Many of these would-be theorists interpret Sir Isaac Newton's work in a way which would have amazed that remarkable genius, and certainly he would not have recognised some of the theory attributed to him.

It may safely be said that owing to the perturbations of tides from meteorological causes, it is possible to find occasions which will yield observational verification of any theory whatever, and generally these theorists pick a day when their theory can challenge all others in accuracy!

Not long ago, one of the authors of this Manual wrote on the errors of tide-gauges, and pointed out that one of the effects of friction was to produce flat tops to the records. Shortly afterwards, he received a letter in which the writer said he had discovered how to predict flat-topped curves!

These preliminary remarks may be taken as cautionary, and express the spirit of the authors of this Manual in attempting to prepare a manual for the study of tidal theory. Every effort has been made to present the subject on the basis of well-established truth, and to give a clear exposition not only of what is known, but also of the difficulties of attaining the perfection of knowledge.

1.2. Practice and theory

In many subjects it has been found that practice and theory have been greatly dissociated, so that it is not at all surprising that men have been found utilising a knowledge of the behaviour of tides when they have been quite ignorant as to the root causes. History repeatedly shows that this may even be regarded as normal. Thus, most of us make truly remarkable applications of electricity, and understand something of the laws which must govern our actions relating to it, without understanding anything of the true nature of it, and even those whose knowledge is unexcelled have to admit that in very truth they also but dimly understand the nature and origin of electrical energy. Similarly we shall show that in the course of the centuries much practical knowledge had been accumulated and used by mariners long before theory was envisaged on correct lines. And even now, with all that has been done, we are still very ignorant as to the most fundamental cause of the

tides, the attraction of one mass upon another. "Action at a distance" by the medium of "gravitational forces" only expresses in words something which none as yet profess fully to understand. There are many things obviously following certain laws which we can confidently use, but as concerning the essential nature we are ignorant.

This state of affairs is greatly exemplified in connection with the tides. We know the forces exactly, but are ignorant of their mode of operation. We can measure the tides and reduce their variations to laws, but there is a great hiatus in our knowledge as to the way in which the forces produce the effects we measure.

With regard to the relation between theory and practice, the following quotation from Whewell's "History of the Inductive Sciences" (1837, vol. ii, p. 248 *et seq.*) is very interesting :—

"The course which analogy would have recommended for the cultivation of our knowledge of tides would have been to ascertain, by an analysis of long series of observations, the effects of changes in the time of transit, parallax and declination of the moon, and thus to obtain the laws of phenomena ; and then to proceed to investigate the laws of causation.

"Though this was not the course followed by mathematical theorists, it was really pursued by those who practically calculated tide-tables ; and the application of knowledge to the useful purposes of life being thus separated from the promotion of theory, was naturally treated as a gainful property, and preserved by secrecy . . . Liverpool, London and other places, had their tide-tables constructed by undivulged methods, which methods, in some instances at least, were handed down from father to son for several generations as a family possession, and the publication of new tables, accompanied by a statement of the mode of calculation, was resented as an infringement of the rights of property.

"The mode in which these secret methods were invented was that which we have pointed out : the analysis of a considerable series of observations. Probably the best example of this was afforded by the Liverpool tide-tables. These were deduced by a clergyman named Holden, from observations made at that port by a harbour-master of the name of Hutchinson, who was led, by a love of such pursuits, to observe the tide for above twenty years, day and night. Holden's tables, founded on four years of these observations, were remarkably accurate.

"At length men of science began to perceive that such calculations were part of their business ; and that they were called upon, as the guardians of the established theory of the universe, to compare it in the greatest possible details with the facts. Mr. Lubbock was the first mathematician who undertook the extensive labours which such a conviction suggested. Finding that regular tide-observations had been made at the London Docks from 1795, he took nineteen years of these (purposely selecting the length of the cycle of the motions of the lunar orbit) and caused them (in 1831) to be analysed by Mr. Dessiou, an expert calculator. He thus obtained tables for the effect of the moon's declination, parallax and the hour of transit, on the tides ; and was enabled to produce tide-tables founded upon the data thus obtained. Some mistakes in these as first published (mistakes unimportant as to the theoretical value of the work) served to show the jealousy of the practical tide-table calculators, by the acrimony with which the oversights were dwelt upon ; but in a very few years, the tables thus produced by an open and scientific process were more exact than those which resulted from any of the secrets ; and thus practice was brought into its proper subordination to theory."

1.3. Qualitative and quantitative theories

Whewell's remarks, quoted in the preceding article, bring out the real importance of the necessity of theory being brought into immediate contact with observations. A historical study of the speculations of men shows that most of the theories prior to the great day of Newton were strictly qualitative in character and not quantitative. Now qualitative theories are very interesting and very useful, but can be highly dangerous, so we shall take the risk of being accused of platitudes by quoting certain pithy remarks by eminent men.

"When you can measure what you are speaking about and express it in numbers, you know something about it, but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind." (Lord Kelvin.)

"A mathematical argument is, after all, only organised common sense, and it is well that men of science should not always expound their work to the few behind a veil of technical language, but should from time to time explain to a larger public the reasoning which lies behind their mathematical notation." (Sir George Darwin.)

Newton's great work on gravitational forces first put tidal theory on a firm foundation, not only because his perspicuity revealed the existence of gravitation in the qualitative sense, and therefore indicated the prime mover in the generation of the tides, but he went further and reduced his theories to calculations which were compared with observations then available to him.

If a theory indicates a phenomenon which has not been observed, then it cannot be regarded as satisfactory until there has been confirmation by observation. Without this safeguard it is very easy to get astray.

In sympathy with the views of Sir George Darwin, every effort has been made in this Manual to reduce the mathematics to its simplest and most elementary forms, and wherever possible to show by general reasoning that the result of mathematical investigation is fundamentally correct.

1.4. Historical review

It is impossible, within the compass of a short chapter, adequately to review the history of a subject like that of tides. It is a subject of very great interest and entertainment, particularly in connection with the speculations of ancient philosophers. Harris's Manual of Tides (United States Coast and Geodetic Survey) gives a very full account of this history, but we shall confine ourselves to progress since the time of Newton.

Sir Isaac Newton (1642–1727), as is well known, discovered the laws of gravitation, and initiated a flow of research into far-reaching gravitational problems. As a result of his investigations, he gave precision to the basis of tidal theory and expressed the variations of forces by means of "the equilibrium tide" which would result under certain idealised conditions. He showed conclusively that the variations of the tides were qualitatively, at least, and to some degree quantitatively, indicated by the variations of the equilibrium tide. Later expositors of the equilibrium tide theory carried it beyond all reasonable interpretation. This Manual devotes special attention to the utilisation of the equilibrium relations, particularly in connection with the Admiralty method of approximate predictions.

It is commonly supposed that Newton's conceptions regarding tides were limited to the equilibrium tide, but this is not doing him justice, for his work was essentially dynamical and he clearly states that the inertia of the water must cause modifications of the simple equilibrium theory.

Some years after the death of Newton, in the year 1738, a prize was offered by the French Academie des Sciences for essays on the tides. We shall only refer to Bernoulli among those who shared the award, partly because he developed the equilibrium theory sufficiently far to give it some practical value when applied with suitable modifications to actual tide observations, but mainly because in this Manual we make extensive use of a theorem in hydraulics which is called Bernoulli's theorem or equation. Just as the equilibrium tide refers to steady conditions, so does Bernoulli's equation, but though it is valid only for motions of fluids in narrow channels and not in broad seas or oceans it has a great range of usefulness. In fact, much of the theory of hydraulics is founded on this equation, and the tidal applications are rendered possible by certain artifices in which the motion is reduced to an apparent steady state.

The famous French scientist Laplace (1749–1827) must be regarded as sharing honours with Newton, for he provided the basis of all modern work on the study of tides considered as fluids in motion. He formulated the equations of motion for tides on a rotating earth, and these equations, of course, include as a particular case

the equation given by Bernoulli. Laplace's equations were solved by him in the case of an ocean covering the whole earth, and it is remarkably to his credit to note that it is only in very recent years that full solutions have been provided for oceans bounded by meridians, though solutions of a formal type (mathematical in nature and not reduced to calculation) have been known for a somewhat longer time.

Laplace was the first to separate tidal phenomena according to species of tide (that is, he dealt separately with the semidiurnal, diurnal and long-period tides), but his perspicuity enabled him to state very clearly the basis of the harmonic method later established by Lord Kelvin, and in fact he formulated a principle as follows :—

" The state of a system of bodies, in which the primitive conditions of the motion have disappeared by the resistances it suffers, is periodical, like the forces which act upon it."

He proceeds to say,

" From this, I concluded that if the sea is actuated by a periodic force expressed by the cosine of an angle which increases uniformly with the time, there results from it a partial tide expressed . . . in the same manner, but in which the constants involved in the angle and the coefficient of the cosine may be, by virtue of accessory circumstances, very different from the same constants in the expression for the force and can be determined only through observation. The expression for the actions of the sun and moon upon the seas can be developed in a convergent series of similar cosines, whence arise as many partial tides as, by the principle of the co-existence of small oscillations, being added together, constitute the tide which is observed in a port."

This quotation showed that Laplace visualised the possibility of the harmonic method, but he did not proceed further with the development, and his methods for tbe analysis and prediction of tides really use the equilibrium method modified in the sense that he dealt separately with the three species of partial tides, and also in that he applied separate constants and phase lags for the lunar and solar parts.

Sir John Lubbock (1803–1865) and Dr. William Whewell (1784–1866) deserve honourable mention for the part they played in stimulating interest in tides. Lubbock provided non-harmonic methods of analyses and predictions which are not even yet entirely disused, and Whewell was instrumental in directing attention to the propagation of tides in the oceans. Apart from this latter interest, however, their work will find little representation in this Manual.

Sir George Airy (1801–1892) is chiefly noted for his expositions of the theory of tides and for his work in connection with the investigation of waves in canals. His canal-theory of tides has not stood the test of time, and to a certain extent it may even be said that it led to much erroneous thought in connection with the propagation of tides, but the greater part of his work on the motion of tides in canals or estuaries, the effects of barriers, the propagation of tides in shallow water, and the results of friction, is of permanent value, and has led, directly or indirectly, to many of the investigations in this Manual.

Lord Kelvin (1824–1907), previously Sir William Thomson, contributed many papers devoted to the theory of tides, but his name is prominently associated with the harmonic method of analysis and prediction, which he devised about the year 1867 ; also, about the year 1872, he invented the tide-predicting machine, though the first machine was not constructed for practical use until a few years later. In these activities he was greatly aided by Mr. E. Roberts.

About the same period, Professor William Ferrel (1817–1891) published valuable researches on tidal theory, and the harmonic method was independently investigated and developed by him.

The harmonic method was greatly developed by Sir George Darwin (1845–1912), and we owe a great deal to him for his systematisation of the subject. It is a tribute to his work that his notation and to some extent his methods have been generally accepted. Modern methods of analyses may appear to differ greatly from the methods he standardised but essentially they are the logical development of his later methods.

The " Manual of Tides " by Rollin A. Harris (1863–1918) of the United States Coast and Geodetic Survey is a remarkable publication and contains a vast amount

of material, not only as to the history of tidal theories, but also in connection with the methods of harmonic analysis and prediction. He gives many valuable tables. Harris was the first to give prominence to the stationary-wave theory, and the modern theories of tides are indebted to him for the stress he laid on the importance of stationary oscillations, but his own theory for tides in oceans is not acceptable because he leaves out of account the important effects of the rotation of the earth.

Lord Rayleigh (1842–1919) contributed many important papers on tidal phenomena, and in particular he showed how the phenomena associated with progressive waves could be studied by the artifice of the observer supposing himself to be travelling with the wave, so reducing the motion to a steady motion to which Bernoulli's equation could be applied. This artifice has proved to be very powerful and Rayleigh's application has been greatly extended in this Manual so as to yield simple proofs of many important results previously demonstrable only by advanced mathematical methods.

It is impossible to deal with others of a long list of names of men who have contributed stage by stage to the development of tidal theory, and we can also only barely mention the efforts of many who in more humble ways have done their little and done it well, whether in Arctic or Antarctic wastes, at sea or on the shores of all the oceans, and have provided the data which are so vital to the subject.

1.5. Descriptive terms

In popular usage the word *tide* is loosely used to represent any phenomenon associated with the tidal movement; thus the same word is used for the "height" of the tide and the "run" of the tide, whereas technically these two phenomena are best described by separate terms.

Thus we define :—

> *The tide* is a periodical movement in the level of the surface of the sea or ocean, due to periodical forces.
>
> *Tidal streams* are the periodical movements of the water, generally horizontally in the seas or oceans, due to periodical forces.

The use of the word "periodical" excludes casual movements due to winds or earthquakes.

> *Currents* are non-periodical movements of water, generally horizontally, and they are due to many causes, such as different temperatures and prevalent winds. Some may be permanent, others temporary.

Mathematicians and other scientists invariably refer to tidal currents and non-tidal currents or permanent currents, but we shall follow the above descriptions according to Admiralty usage.

> *High-water* of a tide at a place refers to the highest level reached by the water surface in one oscillation.
>
> *Low water* similarly refers to the lowest level in one oscillation.
>
> *Mean sea level* at a place is the average value in all states of the oscillation, and this is taken as equivalent to the level which would have existed in the absence of tidal forces.

In this Manual, particularly for theoretical investigations, the elevations of tide are referred to mean sea level, so that high water elevations are positive and low water elevations are negative. This differs from the navigational practice which refers all elevations to a specified datum well below mean sea level.

> *The range of tide* is the difference of elevation between consecutive high and low waters.
>
> *The depth of water* is measured from the surface downwards and the *mean depth* gives the depth of the bottom from the mean level. Two meanings are attached to this expression, (*a*) the mean depth at a place, and (*b*) the mean depth over the area considered. The latter is generally implied but the context states the meaning of the expression.

A bore or *eagre* is a special tidal phenomena found only in certain estuaries or rivers, and is characterised by a sudden onrush of water on a rising tide.

A seiche is a short-period oscillation occurring in a bay or gulf, analogous to the oscillations of water in a dish, and it is unrelated to actual tides, being a natural oscillation set in motion by casual causes.

Tides are divided into *species:* those with a period of about a day are called *diurnal tides* and those witb a period of about half a day are called *semidiurnal tides* and so on. The *long period species* includes all tidal oscillations with periods ranging from 14 days to 19 years, and any others with periods much in excess of one day. The species of tides are divided into *constituents* and the difference between mean sea level and high water (or low water) of a constituent is its *amplitude.*

The tides *generated* by the external forces are said to be *propagated* when they appear to travel like a progressive wave. The propagation of tides is exhibited or illustrated by maps or diagrams on which *cotidal lines* are drawn. These join places at which high water occurs simultaneously. It is necessary to define cotidal lines very carefully as they are sometimes used to denote high waters corresponding to particular astronomical conditions and sometimes they give the high waters of the lunar part of one species of tide. *Co-range lines* join places having equal ranges.

Certain systems of cotidal lines meet in a point, called an *amphidromic point* because of the apparent progression of the tide round the point, at which the range of tide is zero. The system of cotidal lines is called an *amphidromic system.*

Tidal streams may be described by *stream diagrams* which give the direction and velocity of stream at regular intervals of time. When the direction of stream changes through four right angles during a complete cycle, the stream is said to be *rotatory.* When the stream runs to and fro in opposite directions only, it is said to be *rectilinear.*

The velocity of a tidal stream is expressed either in feet per second or in *knots,* and a knot is a speed of one nautical mile per hour. A *nautical mile* is the length of one minute of arc of a meridian, and it varies slightly in different latitudes; the value accepted by the Admiralty is 6080 ft., whereas a statute mile is 5280 ft.

In tbis Manual we shall use nautical miles for distances across seas or along channels.

A *channel* may or may not have vertical sides, uniform breadth and depth, throughout its length; a *canal* is a channel with vertical sides, uniform breadth and deptb, throughout its length.

In theoretical work certain formulæ involve the *coefficient of gravitational force,* which is called the *acceleration due to gravity,* and it is denoted by g. Its value varies with position on a non-spherical earth but tidal theory takes little or no account of variations of the earth's radius, so that a mean value will be taken for g equal to

$$32{\cdot}1722 \text{ ft. per second per second} \quad . \quad . \quad . \quad (1.5a)$$

CHAPTER II

THE TIDE-GENERATING FORCES

2.1. The forces exerted upon a particle

THOUGH it is a relatively simple matter to show that there must be tides upon the earth, yet the explanation of tides as they actually exist must be exceedingly intricate. In addition to an accurate knowledge of the external forces giving rise to tides, it would be necessary to have an accurate knowledge of the parts played by the configuration of the land masses of the earth's surface, by the depths of the oceans, by the rotation of the earth, by the internal forces set up on disturbing the water surface from its normal level, and also by the frictional forces inevitably experienced with moving matter. Such an explanation is quite beyond our present attainments, and therefore no regard should be paid to crude theories purporting

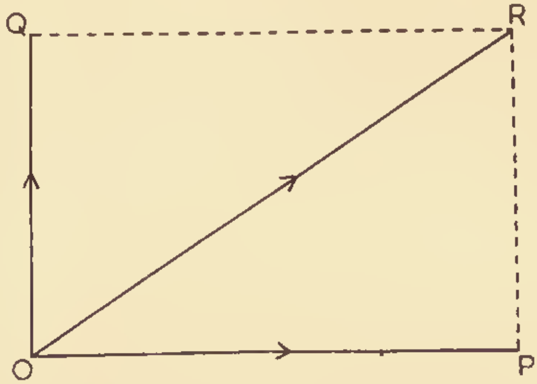

FIG. 2.1. Resolution of velocities and forces.

to set forth in one simple argument the response of the whole mass of the fluid to the tide-generating forces.

Clearly, however, the tide-generating forces are of great importance, and it is very desirable to study their variations in detail, as the characteristics of the tides must have some similarity with the characteristics of the generating forces.

Fortunately, it is possible to specify exactly the forces exerted upon a small portion (or particle) of the fluid by the sun and moon. These forces depend only upon the attraction exerted by one particle of matter upon another, and the laws of action of such forces are well known. The attractive force between two particles varies with the product of the masses (m and m') of the two particles and inversely as the square of the distance (r) between them, so that the attractive force is proportional to mm'/r^2.

This force is exerted along the straight line joining the centres of the two particles, so that we have to consider the direction as well as the magnitude of the force. When dealing with quantities which vary in direction as well as magnitude, it is very convenient to consider *components* as follows. Suppose, for instance, that a vessel is steaming east; relatively to still water she is travelling in a direction parallel to OP in Fig. 2.1, and we shall suppose that her speed is such that she would travel in one minute through a distance represented by the length of OP. If, however, the water itself is moving north, parallel to OQ at right angles to OP, with a velocity which would carry a particle of it in one minute through a distance represented by the length OQ, then the vessel actually travels in the direction OR, and the

resultant velocity is represented in direction and in magnitude by OR. The velocities represented by OP and OQ are said to be the *component velocities*. Then, since

$$\mathrm{OP/OR} = \cos \mathrm{POR}, \quad \mathrm{OQ/OR} = \sin \mathrm{POR},$$

we say that the component velocities of the velocity V in the directions OP and OQ are respectively V cos POR and V sin POR.

Similar reasoning applies to forces, which can be resolved into component forces in any desired directions, though it is most usual to take the directions at right angles to one another, as will be done throughout this chapter.

2.2. The lunar attractive forces

Referring to Fig. 2.2, we shall denote by

O, O′ the centres of the earth and moon, respectively ;
U, U′ the points in which respectively O′O, and O′O produced cut the earth's surface ;

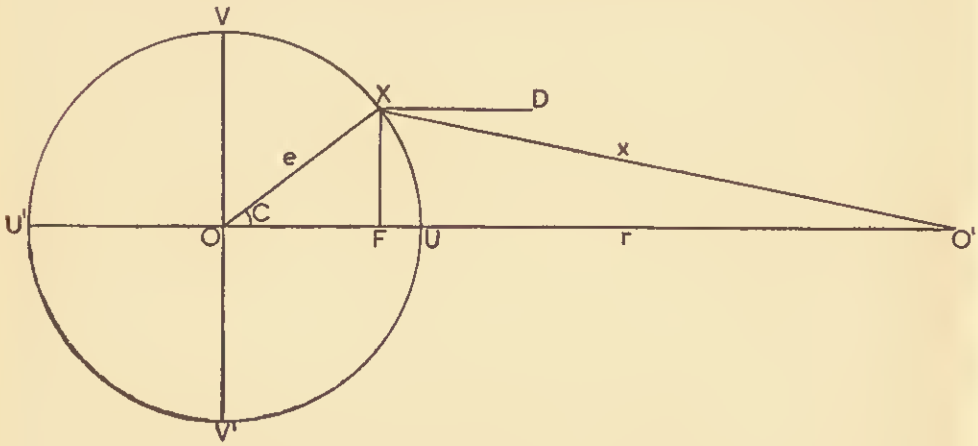

Fig. 2.2. Resolution of attractive forces.

X a point in the great circle in which any plane through OO′ cuts the surface of the earth ;
V, V′ points in the same great circle in quadrature with U, U′
r the distance of the moon from the earth (OO′) ;
e the radius of the earth (supposed spherical) ;
x the distance of X from the moon (XO′) ;
C the angle XOO′.

We shall also denote by

E the mass of the earth ;
M the mass of the moon ;
g the coefficient of gravitational force ;

and we shall draw lines

XD parallel to OO′ ;
XF at right angles to OO′.

Now since g is the attractive force exerted by the earth on a particle of unit mass at the point X, then the attractive force of the moon on this same particle will be less in the ratio of the masses M and E, and less again in the ratio of the squares of the distances from X to the centres of the earth and moon. That is, the attractive force of the moon on a particle of unit mass at X is given by

$$g \frac{\mathrm{M}}{\mathrm{E}} \frac{e^2}{x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.2a)$$

The attractive force of the moon upon a particle of unit mass at the centre of the earth is similarly given by

$$g\frac{M}{E}\frac{e^2}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.2b)$$

Now, if every particle of the earth experienced exactly the same lunar attractive force, both in direction and magnitude, there would be no forces causing relative movements. It is the variation in attractive force which causes one particle to move relatively to another. The average force is clearly equal to that exerted at an average position which we can take as the centre of the earth within the degree of accuracy we require.

The tide-generating force at a point X is therefore defined as the difference in the attractive force experienced at X and that experienced at the centre of the earth. We shall also refer to this as **the differential force.**

We are not interested in the value of the average force referred to above, as it has no tide-raising effects, but it may be well to state that this force would have to be considered if we were studying the forces which keep the moon in its orbit relatively to the earth.

2.3. The lunar tide-generating (or differential) forces

The computation of the differential force for any point X is best effected by mathematical methods, but a general idea of the magnitude and direction of the force can be obtained from a consideration of the forces at four points in Fig. 2.2, viz., U and U′ on the line of centres, the former being directly under the moon, and V and V′ in quadrature with the moon.

We shall take

$$E/M = 81{\cdot}53 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.3a)$$

and if we define

$$c = \text{the mean distance of the moon}$$

we have the ratio

$$c/e = 60{\cdot}26 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.3b)$$

Hence, at U the value of x/e is 59·26 and at U′ it is 61·26. Substituting these in the expression for the attractive force, as given in the previous article, gives the following results :—

$$\left.\begin{array}{l}\text{mean attractive force at U } = 0{\cdot}000{,}003{,}493\,g\\ \text{mean attractive force at O } = 0{\cdot}000{,}003{,}378\,g\\ \text{mean attractive force at U}' = 0{\cdot}000{,}003{,}268\,g\end{array}\right\} \quad . \quad . \quad (2.3c)$$

These attractive forces, of course, are directed towards the moon, and they can be diagrammatically represented as follows, though we have exaggerated the differences between the forces :—

the force at U $\xrightarrow{\hspace{4cm}}$
the force at O $\xrightarrow{\hspace{3cm}}$
the force at U′ $\xrightarrow{\hspace{2cm}}$
the differential force at U $\longrightarrow$
the differential force at U′ $\longleftarrow$

The differential force, as we have defined it, is the difference between the attractive force at the point in question, and the attractive force at the centre of the earth. Now it is clear that according to this definition, the differential force at U, under the moon, will be in the same direction as the attractive forces, whereas that at U′ will have a negative direction, since the force at U′ is less than that at O. It is clear that our definition of the differential force at any point gives a force which must be added to the force at O to give the force at the point, and the above diagram clearly shows that at U′ we must have a force in the opposite direction to that at U.

Hence the lunar differential forces at U and U′ are directed away from the centre of the earth, and the mean values of the differential forces are

$$0{\cdot}000{,}000{,}115\,g \text{ and } 0{\cdot}000{,}000{,}110\,g\text{, respectively.} \quad . \quad . \quad (2.3d)$$

In the plane through V and V′, at right angles to the plane of the paper, all points are practically at equal distances from the moon, and therefore the resultant differential force towards the moon is zero. This is true so far as movement out of the plane is concerned, but there is a slight compressional force in the plane, towards the centre of the earth, because the directions of the attractive forces are inclined slightly to the line of centres. Hence, the attractive force at V is equal to the magnitude of the force at O, multiplied by the cosine of the angle OVO′, which is approximately equal to e/r. If we divide the attractive force at O by the mean value (60·26) of r/e, we get

$$\text{the differential force at V} = 0{\cdot}000{,}000{,}056\,g \qquad . \qquad . \quad (2.3\text{e})$$

since there is no force at O in the plane through V and V′.

Again, therefore, we have differential forces at right angles to the surface of the earth, this time being directed inwards at all points on the plane through V and V′ at right angles to OO′. The results are shown graphically in Fig. 2.3.

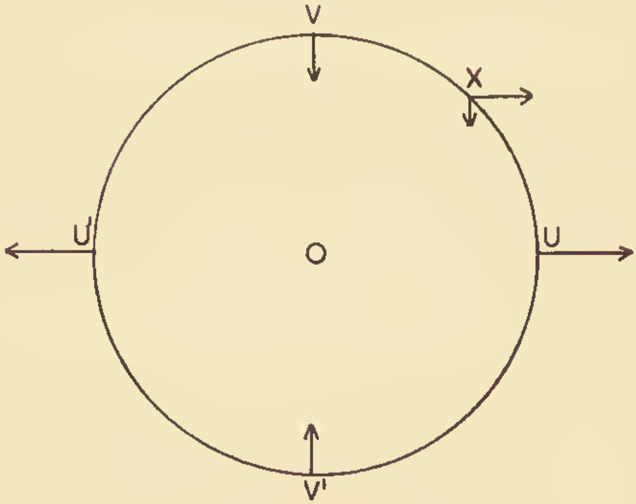

Fig. 2.3. The differential forces.

For points between U and V the forces have components which are intermediate between the component forces at U and V. We shall take the components parallel to the line of centres and at right angles inwards to the line of centres. At U the differential force has a zero component in the latter direction and at V there is a zero component parallel to the line of centres. Hence, as we proceed from U to V the component parallel to the line of centres will continually decrease to zero, while the transverse component will continually increase from zero to the value pertaining to the point V.

Now the differential forces at U, U′, V, V′ are all vertical to the earth's surface, while at the point X, the two components of force, along and transverse to the line of centres, can each be resolved into other components, one along the surface of the earth and one in the vertical direction, to or from the centre of the earth. While it is clear that the horizontal component of differential force will tend to draw a particle along the surface of the earth towards the line of centres it may not be equally clear that the vertical component of differential forces has no such power. We can consider three forces as acting on a particle at the surface as in the figure, viz., the force of gravity g, the vertical component of differential forces which we shall denote for the moment by v, and the horizontal component of differential force which we shall call h. In effect, therefore, we have a resultant force with two components, $g + v$ vertically, and h horizontally, and the resultant force will make an angle with the vertical such that its tangent will be $h/(g + v)$.

Since a fluid tends to set itself with its free surface at right angles to the resultant force, it follows that if the fluid is allowed time to attain this equilibrium condition its surface will be inclined to the original undisturbed surface by an angle whose tangent is $h/(g+v)$. We have already seen that the values of v and h are of the order of one ten-millionth part of g, so that the vertical component v only affects the ultimate result by one-ten-millionth part.

We shall therefore regard the vertical components of differential force as having no practical importance in connection with the generation of tides. It is quite certain that we must look to the tractive power of the horizontal components of differential force for the production of tidal movements, and this conception of the horizontal component being the main agent in the generation of tides is so important

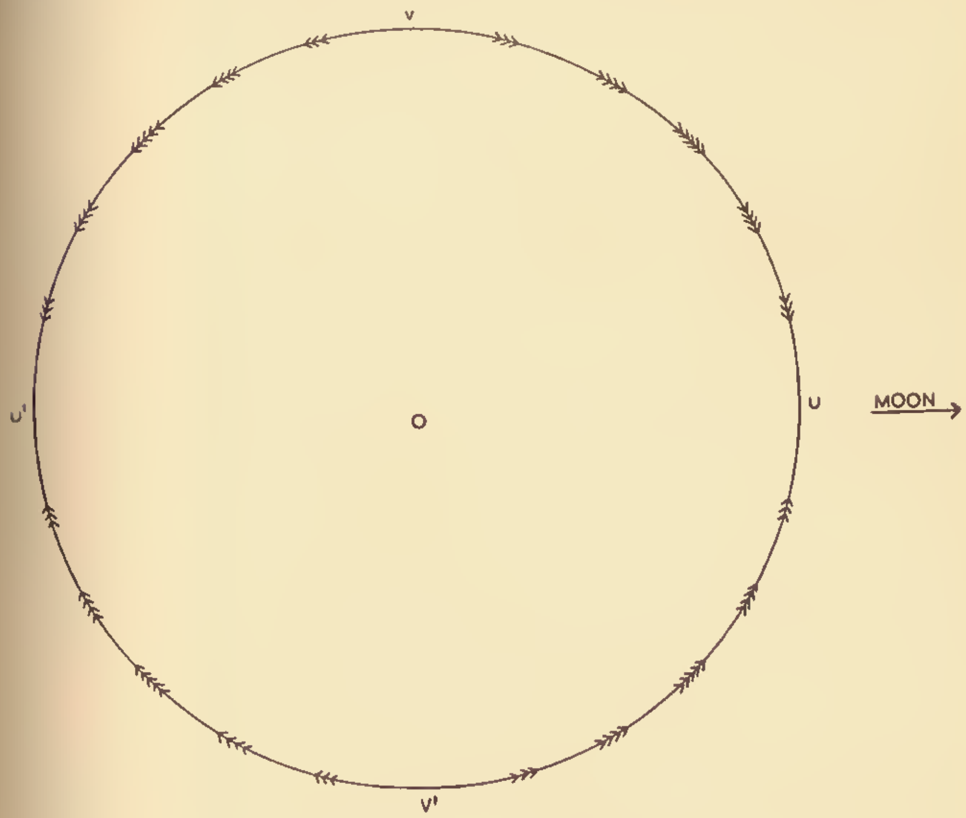

FIG. 2.4. The tractive forces.

that we shall find it convenient to call it the **tractive force** because of its power to move particles along the surface.

The effect of the vertical component of force is interesting in one respect, in that it slightly alters the apparent gravitational force. In other words, a body loses one ten-millionth part of its mean weight when it is directly "under" the moon, so that a ship of 30,000 tons displacement will lose 7 lbs. of its mean weight when the moon is overhead.

We see, therefore, that the tractive force is zero at the points U, U′, V, V′, and that at some point intermediate between U and V it will reach a maximum, directed towards the line of centres or towards the moon but along the surface of the earth. At points intermediate between V and U′ and between U′ and V′ the direction of the force is away from the moon. The directions and magnitudes of the tractive forces are given graphically in Fig. 2.4, where the size of the arrows indicates approximately the relative sizes of the forces.

We have so far only considered a single plane through the line of centres, but if we tried to picture the directions and magnitudes of the forces as experienced over the earth's surface we should get a diagram such as Fig. 2.5. An accurate representation of these forces will be discussed later—our present purpose being to give some general indications of the way in which the tractive forces vary with the position on the earth.

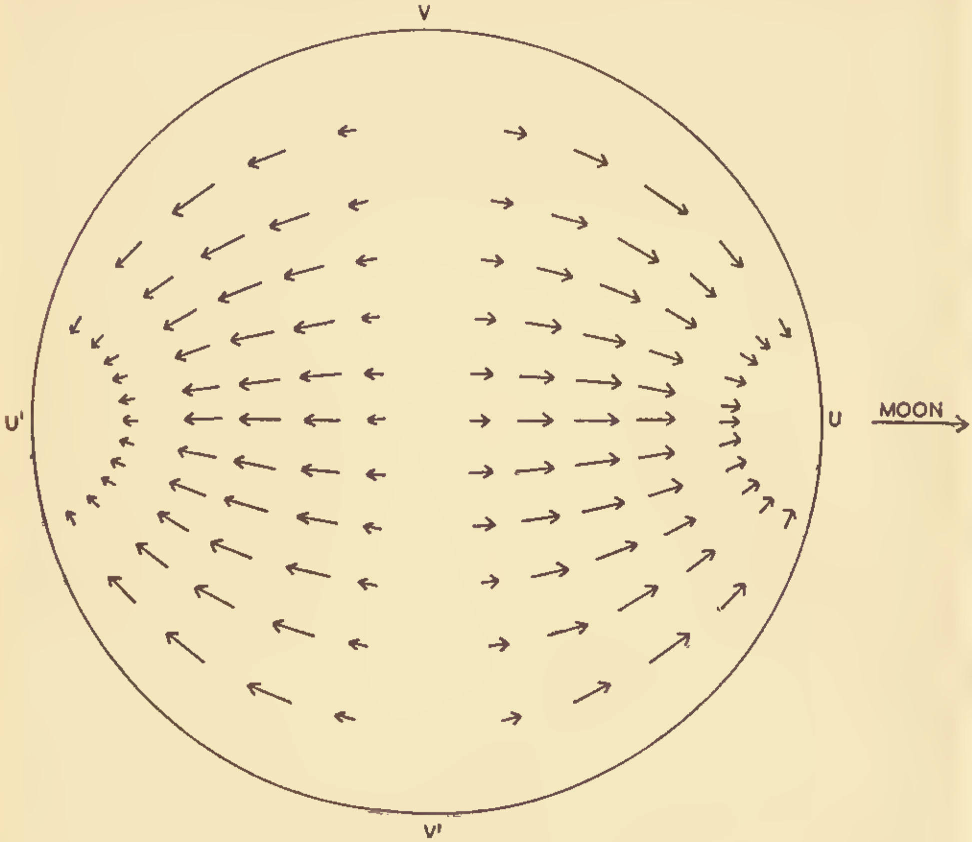

FIG. 2.5. The tractive forces over the earth's surface.

2.4. Formulæ for components of tide-generating (or differential) forces

The application of mathematical methods to the derivation of formulæ for the tractive forces will be postponed to Art. 2.6, and the general reader will probably be content to verify that the formulæ there obtained are in conformity with the conclusions arrived at in Art. 2.3.

The *vertical component* of the lunar differential force (see Art. 2.6) can be expressed by the formula

$$g\,\frac{\mathrm{M}}{\mathrm{E}}\,\frac{e^3}{r^3}\,(2\cos^2\mathrm{C} - \sin^2\mathrm{C}) \quad . \quad . \quad . \quad . \quad (2.4a)$$

where the angle C is the angle XOO′ in Fig. 2.2. As particular cases of this formula we can take the moon at its mean distance as in the previous article, and so obtain the values

$$\begin{aligned} &0{\cdot}000{,}000{,}112\,g \text{ with C} = 0^\circ \text{ or } 180^\circ, \text{ at U and U}' \\ -\,&0{\cdot}000{,}000{,}056\,g \text{ with C} = 90^\circ \text{ or } 270^\circ, \text{ at V and V}' \end{aligned} \quad . \quad . \quad (2.4b)$$

The slight differences between the values of the vertical components at U and U′ will be ignored. These results are clearly in agreement with the direct computations

of the previous paragraph. The positive sign is given for components vertically upwards, and the negative sign for components vertically downwards, towards the centre of the earth.

The *horizontal component* of the lunar differential force is the tractive force and can be expressed (see Art. 2.6) by the formula

$$\frac{3}{2} g \frac{M}{E} \frac{e^3}{r^3} \sin 2C \quad . \quad . \quad . \quad . \quad . \quad (2.4c)$$

In conformity with previous deductions, we see that the formula gives zero values of tractive force when $C = 0°$, $90°$, $180°$ or $270°$, and the direction of the force is positive in the clockwise direction when C lies between 0° and 90°, or between 180° and 270°, and it is in the reverse direction in the other quadrants. We have no direct computation of the magnitudes of the tractive forces in the previous article, but it is clear that they must be of the same order as those found for the vertical components. With the moon at its mean distance, the maximum value of the tractive force is

$$0{\cdot}000{,}000{,}084\, g \quad . \quad . \quad . \quad . \quad . \quad (2.4d)$$

at points exactly half-way between U, V, U′ and V′, this again being in conformity with previous deductions.

It should be noted that whereas the direct attractive force exerted by the moon on a particle depends on the inverse square of the moon's distance, the differential forces and their components vary inversely with the cube of the moon's distance, or directly with the cube of the moon's parallax. This important result will be frequently referred to in later theory.

The components of the *solar forces* are obtained in precisely the same manner as those of the lunar forces, and it is only necessary to replace the mass of the moon (M) and its distance (r), by the mass of the sun (S) and its distance (r'). The ratio of the masses of the sun and moon is about 27,100,000 and the ratio of their mean distances is about 389, but though the mass of the sun is so very much greater than that of the moon, yet the cube of the ratio of distances is so great that the solar forces are on the average only about 0·460 times those of the moon.

2.5. The genesis of tidal streams

It is of some interest to obtain an estimate of the movement of a free particle on the supposition that it is unaffected by the movements of other particles. From what has been said, it will be understood that this state is very far removed from reality.

Taking the moon at its mean distance, it has been stated that the maximum tractive force on a particle of unit mass has the value

$$0{\cdot}000{,}000{,}084\, g \quad . \quad . \quad . \quad . \quad (\text{see } 2.4d)$$

If this continued unchanged through the time t it would cause the particle to acquire a velocity

$$0{\cdot}000{,}000{,}084\, gt \quad . \quad . \quad . \quad . \quad . \quad (2.5a)$$

so that in one hour ($t = 60 \times 60$sec) the acquired velocity would be

$$\begin{aligned} & 0{\cdot}000{,}000{,}084 \times 32 \times 60 \times 60 \text{ ft. per second} \\ &= 0{\cdot}000{,}000{,}084 \times 32 \times (60 \times 60)^2 \text{ ft. per hour} \\ &= 35 \text{ ft. per hour} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.5b) \end{aligned}$$

If we suppose that the tidal period is 12 hours and that the velocity changes in the same way as a sine curve, so that the velocity at hour 0 is zero, then at hour 1 it would be rather less than 35 ft. per hour, and a sine curve will show that the maximum velocity will be twice this value, that is, 70 ft. per hour. Repeating this process, commencing with the maximum velocity, in one hour the distance travelled would be 70 ft. and the maximum excursion of the particle from its mean position would be about 140 ft.

Tides, as they exist upon the earth, may be considered as having reached a steady state of pulsation, so that we do not need to attempt to follow the processes of generating the tides from a state in which the fluid is everywhere motionless.

But we note that if the fluid were motionless the tractive forces would immediately cause tidal streams and that these would produce elevation of the fluid surface. If an elevation is formed, however, the back pressure of the water will tend to reduce the streams to rest and then to reverse them, so that actual tides and tidal streams are in a state of interaction in which the parts respectively played are of equal importance. In fact, on the earth as we know it, tides cannot exist in the absence of tidal streams.

While it is true that tidal streams are more prominent in the relatively shallow waters near the coasts, it is a fallacy to conclude, as many have done in the past, that there are no tidal streams in the deep oceans. The tidal streams exist though they may be ordinarily imperceptible owing to the necessary transport of water being effected through a larger vertical area than is the case near the coasts.

An estimation of the elevation of the surface cannot be given at this stage, for the elevation depends also upon the configuration of the ocean, and in fact such an attempt would result in far more importance being attached to the above calculation than would be warranted by the assumptions made. The object of the investigation given above was simply to show that a direct effect of the tractive forces is to produce tidal streams even in very deep water, and that it is a fallacy to assume that tidal streams only occur in coastal waters, or that they only come into existence when the depth of water decreases suddenly.

It should be carefully noted that the depth of the ocean being small relatively to the radius of the earth, then the variations of the tractive forces tend to produce tidal streams which have the same magnitude and direction from top to bottom. We shall show in later chapters that this is true in general if frictional forces are left out of consideration.

*2.6. Mathematical investigation of the tide-generating (or differential) forces

The attractive force of the moon at the point X in Fig. 2.2 has been shown to have a magnitude of

$$g \frac{M}{E} \frac{e^2}{x^2} \qquad \text{(see 2.2a)}$$

and to be directed along XO′. This force can be resolved into components

$$g \frac{M}{E} \frac{e^2}{x^2} \cos \mathrm{DXO'} \text{ along XD, parallel to OO'} \qquad (2.6a)$$

and

$$g \frac{M}{E} \frac{e^2}{x^2} \sin \mathrm{DXO'} \text{ along XF, at right angles to OO'} \qquad (2{\cdot}6b)$$

If the angle XOO′ is denoted by C, then

$$\mathrm{FO'} = \mathrm{OO'} - \mathrm{OF} = r - e \cos \mathrm{C}$$
$$\mathrm{XF} = e \sin \mathrm{C}$$

while the angle DXO′ is equal to the angle XO′F.

Hence,

$$\cos \mathrm{DXO'} = \cos \mathrm{XO'F} = \mathrm{FO'/XO'} = (r - e \cos \mathrm{C})/x$$
$$\sin \mathrm{DXO'} = \sin \mathrm{XO'F} = (e \sin \mathrm{C})/x,$$

and therefore the two component forces become

$$g \frac{M}{E} \frac{e^2}{x^2} \frac{(r - e \cos \mathrm{C})}{x} \text{ parallel to OO'} \qquad (2.6c)$$

and

$$g \frac{M}{E} \frac{e^2}{x^2} \frac{e \sin \mathrm{C}}{x} \text{ at right angles to OO'} \qquad (2.6d)$$

The force at the centre of the earth O was shown to be

$$g \frac{M}{E} \frac{e^2}{r^2}$$

* See par. 1, page vii.

directed along OO′, so that if we take the differential force (that is, the difference between the force at X and that at O) we can subtract the components of the latter from those of the former, giving

$$g\frac{\mathrm{M}}{\mathrm{E}}\left\{\frac{e^2}{x^2}\frac{(r-e\cos \mathrm{C})}{x}-\frac{e^2}{r^2}\right\} \text{ parallel to OO}' \quad . \quad . \quad . \quad (2.6e)$$

and

$$g\frac{\mathrm{M}}{\mathrm{E}}\frac{e^2}{x^2}\frac{e\sin \mathrm{C}}{x} \qquad \text{at right angles to OO}' \quad . \quad . \quad (2.6f)$$

We see that the force at O has no component at right angles to OO′ so that (2.6d) has remained unchanged.

The expression (2.6e) can be arranged to give

$$g\frac{\mathrm{M}}{\mathrm{E}}\left\{\frac{e^2}{x^2}\frac{r}{x}-\frac{e^2}{r^2}-\frac{e^3\cos \mathrm{C}}{x^3}\right\}$$

$$=g\frac{\mathrm{M}}{\mathrm{E}}\left\{\frac{e^2(r^3-x^3)}{x^3r^2}-\frac{e^3\cos \mathrm{C}}{x^3}\right\}$$

$$=g\frac{\mathrm{M}}{\mathrm{E}}\left\{\frac{e^2(r-x)(r^2+rx+x^2)}{x^3r^2}-\frac{e^3\cos \mathrm{C}}{x^3}\right\} \quad . \quad . \quad (2.6g)$$

Now, from Fig. 2.2, we see that if the distance of the moon is very much greater than the radius of the earth, then the angle XO′F is very small, so that XO′ is practically equal to FO′, whence we get, within practical limits,

$$r-x=\mathrm{OO}'-\mathrm{FO}'=\mathrm{OF}=e\cos \mathrm{C}$$

so that the expression (2.6 g) becomes equal to

$$g\frac{\mathrm{M}}{\mathrm{E}}\frac{e^3\cos \mathrm{C}}{x^3}\left\{\frac{r^2+rx+x^2}{r^2}-1\right\}$$

$$=g\frac{\mathrm{M}}{\mathrm{E}}\frac{e^3\cos \mathrm{C}}{x^3}\left\{\frac{rx+x^2}{r^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.6h)$$

Again, because the value of r is about 60 times the value of e, it follows that the relative variations of the factor

$$\frac{rx+x^2}{r^2}$$

are negligibly small compared with the relative variations in the factor

$$\frac{e^3\cos \mathrm{C}}{x^3}$$

and consequently we can now replace x by r without serious loss, so far as our present purposes are concerned, though exact tidal theory, of course, takes the variation into account. Similarly, for the force in the direction at right angles to OO′, we can now replace x by r in (2.6f).

Hence the two component forces become

$$2g\frac{\mathrm{M}}{\mathrm{E}}\frac{e^3}{r^3}\cos \mathrm{C} \quad \text{in a direction parallel to OO}' \quad . \quad . \quad . \quad (2.6i)$$

and

$$g\frac{\mathrm{M}}{\mathrm{E}}\frac{e^3}{r^3}\sin \mathrm{C} \quad \text{in a direction at right angles to OO}' \quad . \quad (2.6j)$$

In order to determine the components of differential force along the surface of the earth and vertically upwards, we must refer to Fig. 2.6, which represents a small part of Fig. 2.2. Let the components (2.6i) and (2.6j) of the differential force at X be expressed by the directions and lengths of the lines XD and XF respectively. Let XY represent a tangent to the earth's surface and draw DD′ and FF′ at right angles to XY. Then the angle FXY is equal to the angle XOF, and the angle XDD′ is also equal to the angle FXY.

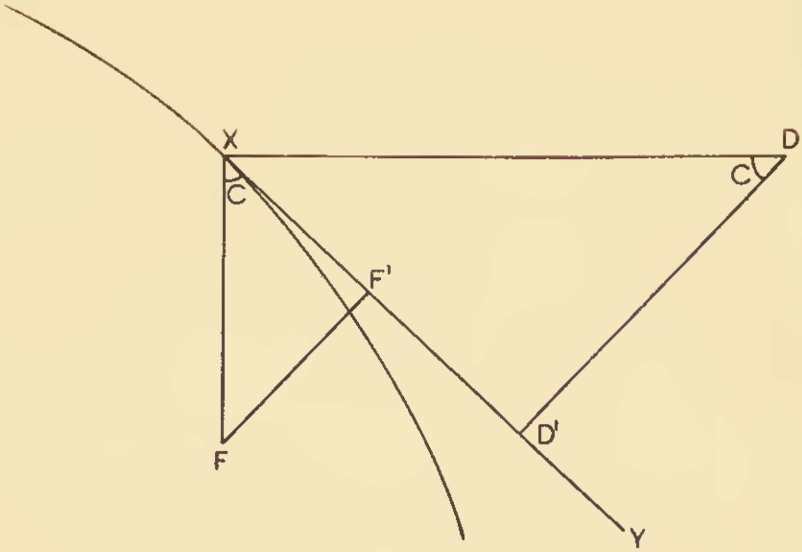

FIG. 2.6. Derivation of tractive forces from components of differential force.

Hence we have XDD′ = FXY = C, and the components of differential force along XY are thus

$$(2.6i) \sin C \quad \text{and} \quad (2.6j) \cos C \qquad . \quad . \quad . \quad . \quad (2.6k)$$

so that the total component force along XY is given by the sum of

$$2\,g\,\frac{M}{E}\frac{e^3}{r^3}\cos C \sin C \quad \text{and} \quad g\,\frac{M}{E}\frac{e^3}{r^3}\sin C \cos C$$

which is equal to

$$\frac{3}{2}\,g\,\frac{M}{E}\frac{e^3}{r^3}\sin 2C \qquad . \quad . \quad . \quad . \quad . \quad (2.6l)$$

and this is the expression for the tractive force along the surface of the earth.

Similarly, the component of differential force vertically upwards is equal to

$$(2{\cdot}6i) \cos C - (2.6j) \sin C \qquad . \quad . \quad . \quad . \quad (2.6m)$$

and this gives the expression

$$g\,\frac{M}{E}\frac{e^3}{r^3}(2\cos^2 C - \sin^2 C) \qquad . \quad . \quad . \quad . \quad (2.6n)$$

The solar tide-generating (or differential) force can be expressed by formulæ similar to those given for the moon.

CHAPTER III

GENERAL DEDUCTIONS FROM THE TRACTIVE FORCES

3.1. Diagrammatic representation of the lunar tractive forces

(In this chapter attention will be concentrated upon the variation of the lunar tractive forces, but with obvious changes of phraseology the discussion also applies to the solar tractive forces.)

WE have so far been mainly concerned with the discussion of points on a circle representing the intersection with the earth of a plane through the centres of the

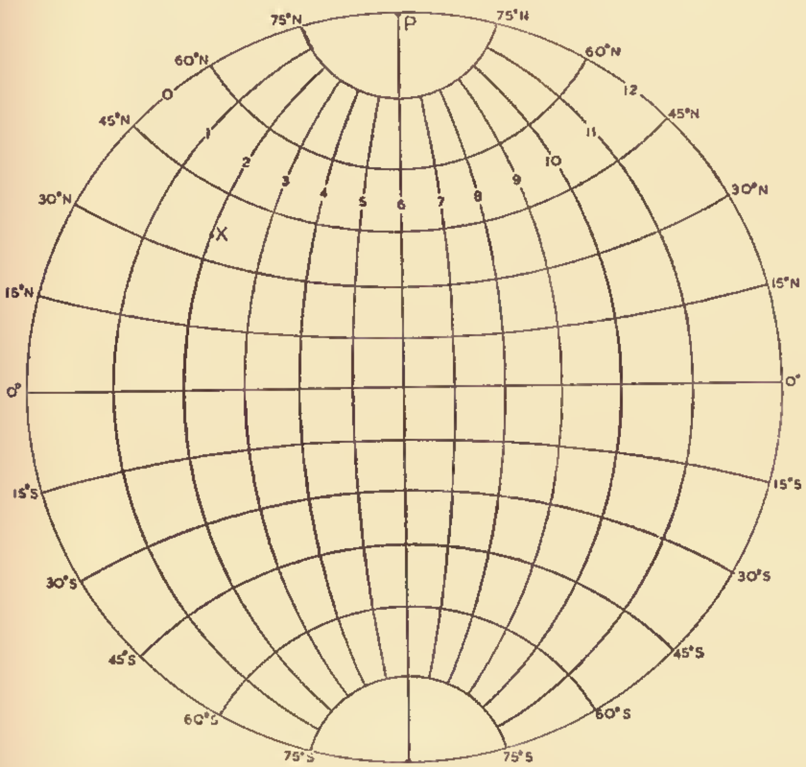

FIG. 3.1. Meridians of longitude and parallels of latitude on stereographic projection.

earth and moon. We now proceed to picture more accurately than in Fig. 2.5 the variation of force over the globe, and thence to deduce methods for finding the variation of the force at a particular point as the earth rotates.

The diagrammatic representation of lines on a curved surface presents many difficulties, so we shall first draw a diagram of parallels of latitude and meridians of longitude, which will familiarise us with the principles involved. In Fig. 3.1 parallels of latitude are shown at intervals of 15° and meridians are shown also at intervals of 15°. (The projection is that known as "stereographic projection," being taken from a point on the central meridian where it cuts the equator. Instructions

for drawing such projections are given in Art. 3.7.) The meridians are numbered from 0 to 12 and the lines may also be considered as the positions of any one meridian at intervals at 1/24th part of the time between successive transits of the moon; that is, the hours are lunar hours. Thus a point X will be on the bounding circle at hour 0, and at the position shown in the diagram at hour 2.

An exactly similar diagram is shown in Fig. 3.2 except that it is turned through 90°, so that the "axis" is turned towards the moon. The principal plane, containing the bounding circle and the centres of the earth and moon, is supposed to pass also through the poles of the earth, but the exact position of the north pole on the bounding

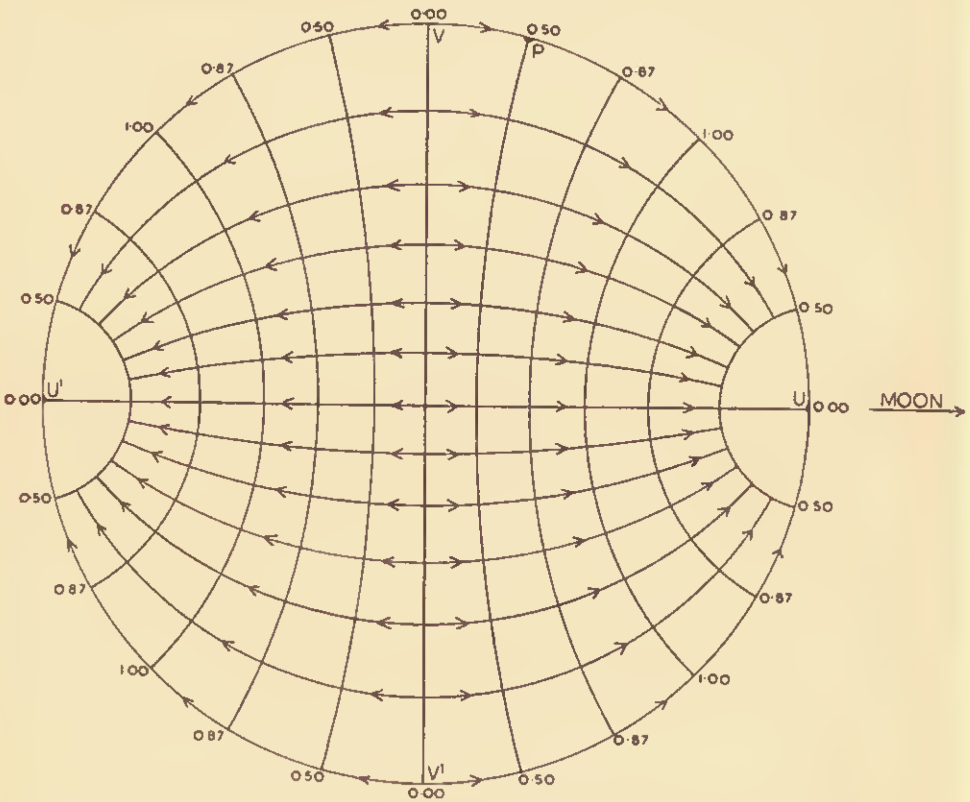

FIG. 3.2. Directions and relative magnitudes of the lunar tractive forces. (P is the position of the north pole of the earth when the moon is 15° N. of the equator.)

circle will depend upon the position of the moon north or south of the equator. In the diagram the north pole P is shown for the case where the moon is 15° N. of the equator.

The parallel circles join points having equal magnitudes of tractive force. The central circle joins all those points at which there is zero tractive force, and so separates the earth's surface into two regions; on one side the force is directed more or less towards the moon but of course on the surface of the earth, while on the other side it is directed away from the moon. The directions of the tractive forces will be along lines akin to the meridians, these being the circles in which different planes through the centres of earth and moon cut the earth's surface. The maximum values of tractive force are denoted by 1·00, but the actual values of the tractive forces are to be found by multiplying the figures on the diagram by

$$\frac{3}{2} g \frac{M}{E} \frac{e^3}{r^3}$$

3.2. Variation of force at a given point, hour by hour

The two diagrams described in the previous article enable us to trace the variations of the tractive force hour by hour for any place on the earth's surface, and for any declination of the moon, that is, the angular distance of the moon from the equator. We shall, however, give illustrations for a declination of 15° N. and for latitudes 0°, 30° and 60° N.

Let a tracing of Fig. 3.2 be placed over Fig. 3.1 so that the equator makes an angle of 15° with the line of centres of the earth and moon. The north pole and the equator will then be in the positions shown in Fig. 3.3. Trace the equator, and the parallels of latitude 30° N. and 60° N. and mark on them the hours 0, 1, 2, . . . 12. The result is shown in Fig. 3.3.

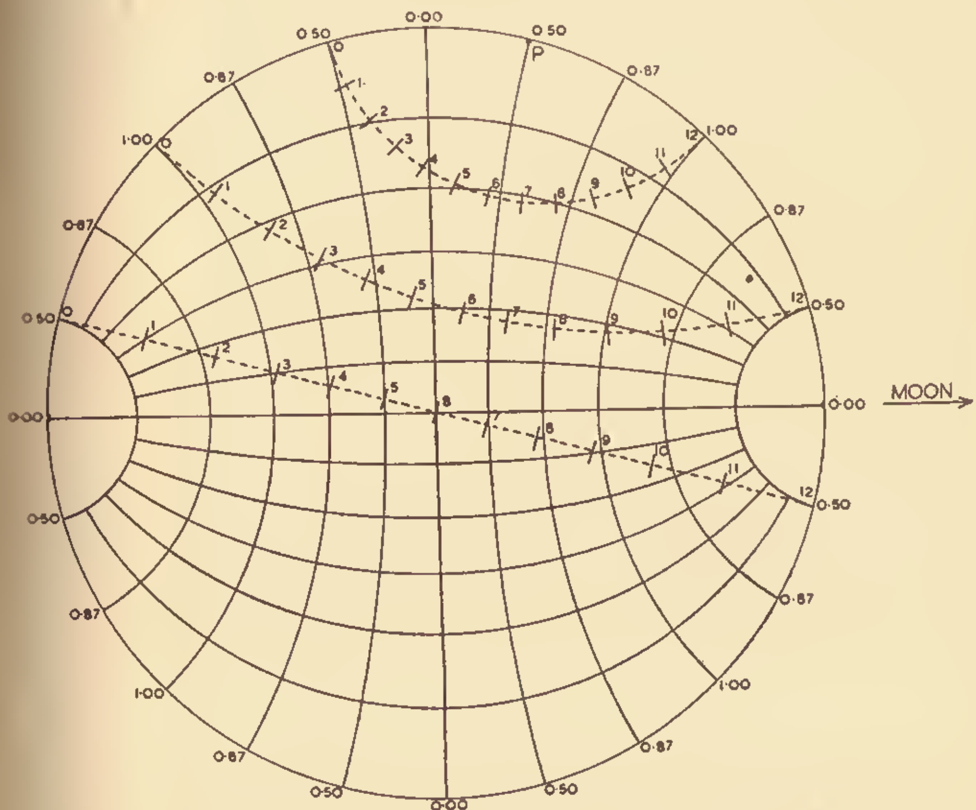

FIG. 3.3. Deduction of variations of lunar tractive forces in certain latitudes, hour by hour with moon 15° N. of the equator. (P is the position of the north pole of the earth.)

There is no distortion of angles by the projection so that we can take angles just as they appear on the drawing; only small portions of the meridians are shown in Fig. 3.3, but they may be traced in full from Fig. 3.1, if desired. Consider the case of latitude 30° N. At hour 0 the force is 1·0 in magnitude and is directed due south, at hour 2 it has a magnitude of about 0·9 and its direction is approximately south-west. At hour about 5·5 the force is zero, and there is a change in direction thereafter from west to east. At hour 9 the force is 1·0 in magnitude and is directed a little to the south of east, and at hour 12, the value is 0·5 and the direction due south.

From this point the same diagram can be used for hours up to 24, and it is readily verified that there are certain symmetrical and asymmetrical relations which can

be used. Thus the magnitude of the force at hour 13 is equal to that at hour 11, but the direction at hour 13 is as much to the west of south as it is to the east of south at hour 11. Similar relations hold for the forces at hours 14 and 10, and so on.

If the diagram is carefully drawn on a large scale, and if the angles are carefully measured, then it is possible to get results such as are given in Table 3.2. Similar tables can be constructed for the variation of tractive forces at points on the equator, as in Table 3.1, or for points in latitudes 60° or 89°·9, as in Tables 3.3 and 3.4, and, of course, for any other desired latitude or for any desired declination. It is, however, very difficult to obtain exact values from a diagram and those given in Tables 3.1 to 3.4 have been computed (see Art. 3.6).

An alternative method of reading the diagram is to take the intersections of the lines giving the directions of the forces with the parallels of latitude, to read the angles, and to tabulate the results against the times corresponding to the intersections.

TABLES 3.1 TO 3.4

Examples of the Variation of Tractive Forces

General factor $\frac{3}{2} g \frac{M}{E} \frac{e^3}{r^3}$

LUNAR DECLINATION 15° N.

Hour	Table 3.1 Equator		Table 3.2 Latitude 30° N.		Table 3.3 Latitude 60° N.		Table 3.4 Latitude 89°·9 N.	
0	0·50	S.	1·00	S.	0·50	S.	0·50	N.
1	0·65	S. 42° W.	1·00	S. 20° W.	0·47	S. 14° W.	0·50	N. 15° E.
2	0·92	S. 62° W.	0·95	S. 37° W.	0·38	S. 29° W.	0·50	N. 30° E.
3	1·00	S. 69° W.	0·82	S. 50° W.	0·24	S. 44° W.	0·50	N. 45° E.
4	0·85	S. 73° W.	0·55	S. 61° W.	0·03	S. 57° W.	0·50	N. 60° E.
5	0·45	S. 73° W.	0·17	S. 69° W.	0·19	N. 70° E.	0·50	N. 75° E.
6	0·00	....	0·26	N. 77° E.	0·44	N. 82° E.	0·50	E.
7	0·45	N. 73° E.	0·65	N. 84° E.	0·62	S. 84° E.	0·50	S. 75° E.
8	0·85	N. 73° E.	0·92	S. 89° E.	0·82	S. 71° E.	0·50	S. 60° E.
9	1·00	N. 69° E.	1·00	S. 80° E.	0·93	S. 56° E.	0·50	S. 45° E.
10	0·92	N. 62° E.	0·89	S. 68° E.	0·98	S. 39° E.	0·50	S. 30° E.
11	0·65	N. 42° E.	0·65	S. 46° E.	1·00	S. 20° E.	0·50	S. 15° E.
12	0·50	N.	0·50	S.	1·00	S.	0·50	S.
13	0·65	N. 42° W.	0·65	S. 46° W.	1·00	S. 20° W.	0·50	S. 15° W.
14	0·92	N. 62° W.	0·89	S. 68° W.	0·98	S. 39° W.	0·50	S. 30° W.
15	1·00	N. 69° W.	1·00	S. 80° W.	0·93	S. 56° W.	0·50	S. 45° W.
16	0·85	N. 73° W.	0·92	S. 89° W.	0·82	S. 71° W.	0·50	S. 60° W.
17	0·45	N. 73° W.	0·65	N. 84° W.	0·62	S. 84° W.	0·50	S. 75° W.
18	0·00	....	0·26	N. 77° W.	0·44	N. 82° W.	0·50	W.
19	0·45	S. 73° E.	0·17	S. 69° E.	0·19	N. 70° W.	0·50	N. 75° W.
20	0·85	S. 73° E.	0·55	S. 61° E.	0·03	S. 57° E.	0·50	N. 60° W
21	1·00	S. 69° E.	0·82	S. 50° E.	0·24	S. 44° E.	0·50	N. 45° W.
22	0·92	S. 62° E.	0·95	S. 37° E.	0·38	S. 29° E.	0·50	N. 30° W.
23	0·65	S. 42° E.	1·00	S. 20° E.	0·47	S. 14° E.	0·50	N. 15° W.
24	0·50	S.	1·00	S.	0·50	S.	0·50	N.

3.3. Necessity for resolution into component forces

If we take the results of the variation of the tractive forces, hour by hour, from Table 3.2, they can be exhibited diagrammatically as in Fig. 3.4. For example, if for latitude 30° N. we take hour 2, and draw from the origin in the direction S. 37° W. and mark off along the line a distance of 0·95 of some convenient unit such as ten centimetres, we get the terminal point marked 2 on the diagram. The variation of force, when exhibited in this way, seems very complicated, and it is obvious that any diagrammatic representation of the complete force must be of little value. If, however, we take the two components of force to the north and east directions we simplify issues very considerably. If this operation is performed, either graphically or by mathematical methods, we obtain the components to the north and east as given in Table 3.6 for the case under consideration. These components can then be graphed as in Fig. 3.6. Similarly, Tables 3.5, 3.7, 3.8 and Figs. 3.5, 3.7, and 3.8 give corresponding results for the cases where the latitude of the given point X is 0°, 60° or 89°·9 (taken instead of 90° because at the pole there cannot be north or east components). In these figures the unbroken lines represent the east components and the pecked lines represent the north components.

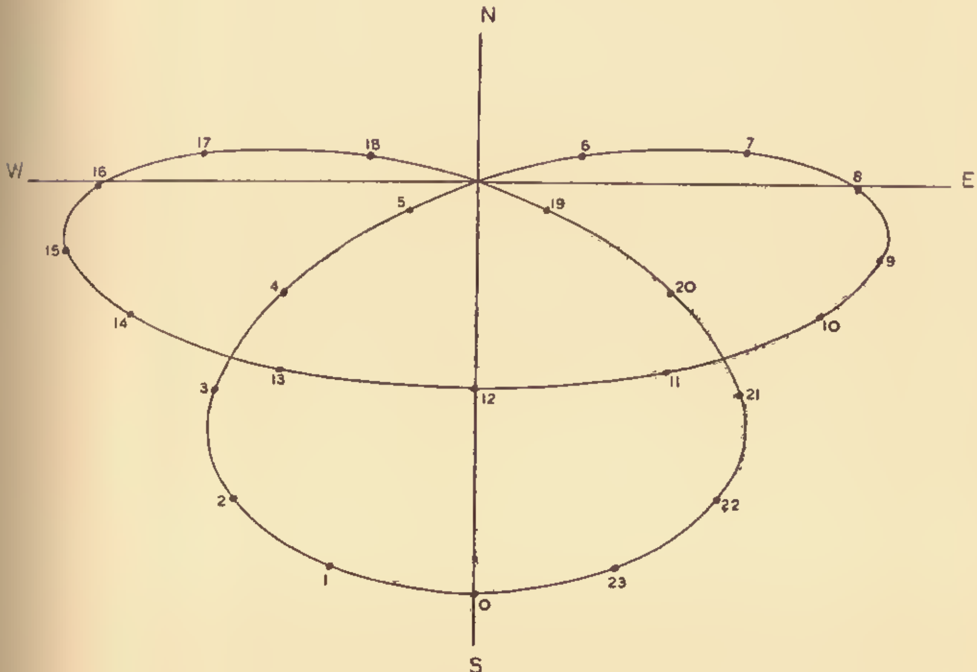

FIG. 3.4. Example of variation of lunar tractive forces, hour by hour. (Latitude 30° N., lunar declination 15° N.)

TABLES 3.5 TO 3.8

North and East Components of Tractive Forces

General factor $\frac{3}{2} g \frac{M}{E} \frac{e^3}{r^3}$

LUNAR DECLINATION 15° N.

Lunar Hour	Table 3.5 Latitude 0° N.	E.	Table 3.6 Latitude 30° N. N.	E.	Table 3.7 Latitude 60° N. N.	E.	Table 3.8 Latitude 89°·9 N. N.	E.
0	− 0·50	0·00	− 1·00	0·00	− 0·50	0·00	0·50	0·00
1	− 0·48	− 0·43	− 0·94	− 0·34	− 0·46	− 0·11	0·48	0·13
2	− 0·43	− 0·81	− 0·77	− 0·58	− 0·33	− 0·19	0·43	0·25
3	− 0·35	− 0·93	− 0·52	− 0·63	− 0·17	− 0·16	0·35	0·35
4	− 0·25	− 0·81	− 0·27	− 0·48	− 0·02	− 0·03	0·25	0·43
5	− 0·13	− 0·43	− 0·06	− 0·16	0·07	0·19	0·13	0·48
6	0·00	0·00	0·06	0·25	0·06	0·43	0·00	0·50
7	0·13	0·43	0·07	0·65	− 0·06	0·65	− 0·13	0·48
8	0·25	0·81	− 0·02	0·92	− 0·27	0·78	− 0·25	0·43
9	0·35	0·93	− 0·17	0·99	− 0·52	0·77	− 0·35	0·35
10	0·43	0·81	− 0·33	0·83	− 0·77	0·62	− 0·43	0·25
11	0·48	0·43	− 0·46	0·47	− 0·94	0·35	− 0·48	0·13
12	0·50	0·00	− 0·50	0·00	− 1·00	0·00	− 0·50	0·00
13	0·48	− 0·43	− 0·46	− 0.47	− 0·94	− 0·35	− 0·48	− 0·13
14	0·43	− 0·81	− 0·33	− 0·83	− 0·77	− 0·62	− 0·43	− 0·25
15	0·35	− 0·93	− 0·17	− 0·99	− 0·52	− 0·77	− 0·35	− 0·35
16	0·25	− 0·81	− 0·02	− 0·92	− 0·27	− 0·78	− 0·25	− 0·43
17	0·13	− 0·43	0·07	− 0·65	− 0·06	− 0·65	− 0·13	− 0·48
18	0·00	0·00	0·06	− 0·25	0·06	− 0·43	0·00	− 0·50
19	− 0·13	0·43	− 0·06	0·16	0·07	− 0·19	0·13	− 0·48
20	− 0·25	0·81	− 0·27	0·48	− 0·02	0·03	0·25	− 0·43
21	− 0·35	0·93	− 0·52	0·63	− 0·17	0·16	0·35	− 0·35
22	− 0·43	0·81	− 0·77	0·58	− 0·33	0·19	0·43	− 0·25
23	− 0·48	0·43	− 0·94	0·34	− 0·46	0·11	0·48	− 0·13
24	− 0·50	0·00	− 1.00	0·00	− 0·50	0·00	0·50	0·00

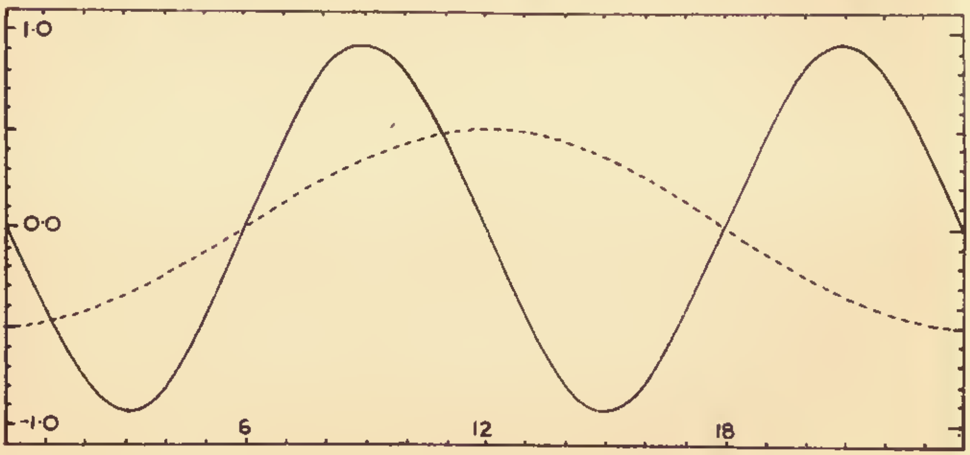

FIG. 3.5. Components of lunar tractive forces.
(Latitude 0°, lunar declination 15° N.)

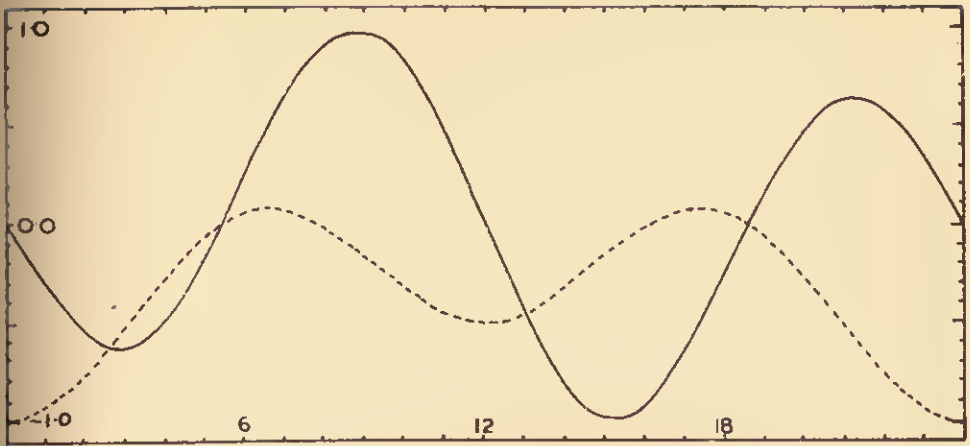

FIG. 3.6. Components of lunar tractive forces.
(Latitude 30° N., lunar declination 15° N.)

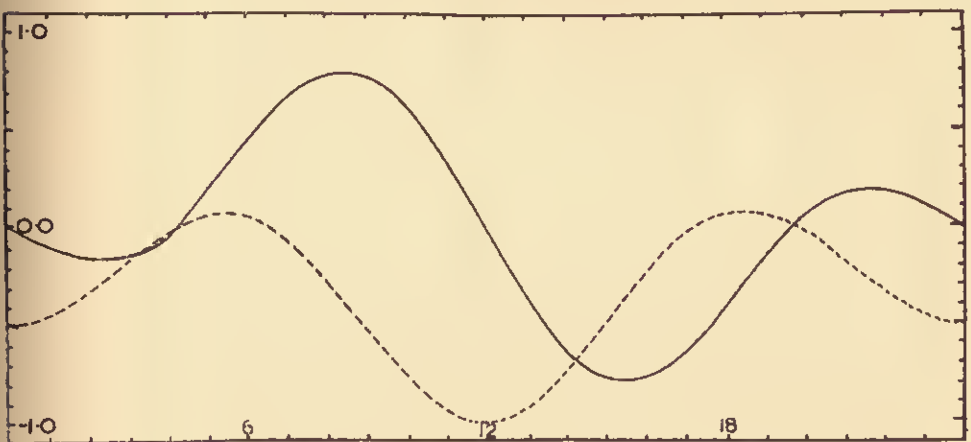

FIG. 3.7. Components of lunar tractive forces.
(Latitude 60° N., lunar declination 15° N.)

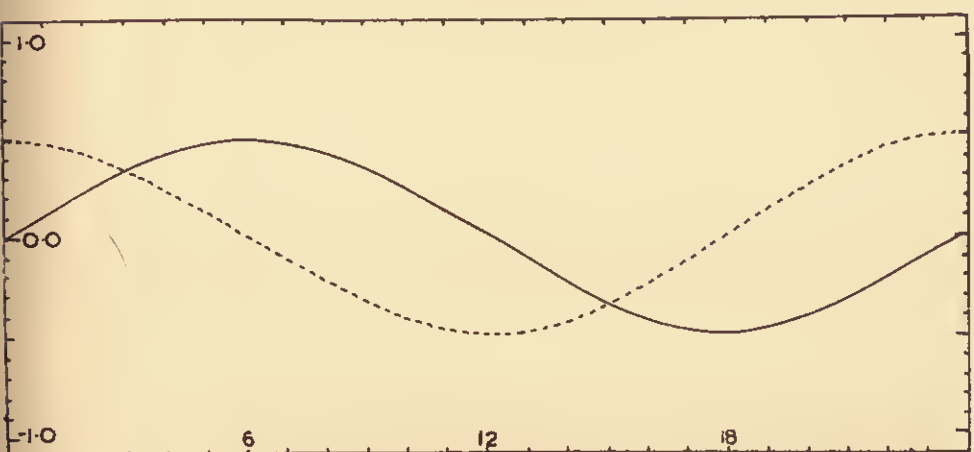

FIG. 3.8. Components of lunar tractive forces.
(Latitude 89°·9 N., lunar declination 15° N.)

It is obvious that some of these curves are simpler than the others. Thus the east component for latitude 0° is a regular oscillation which repeats itself after 12 lunar hours, and it is therefore called a *semidiurnal oscillation*. For latitude 89°·9 N. the east component of tractive force again manifests a simple type of variation, with a period of 24 lunar hours, and it is therefore called a *diurnal oscillation*. For intermediate latitudes, the oscillation is obviously *mixed* in that there are evidences of both diurnal and semidiurnal oscillations. Hence we conclude that for the east component of force, the diurnal part increases, and the semidiurnal part decreases, from the equator to the pole.

For the north components, we note that there is a pure diurnal oscillation for points on the equator and also near the pole, but that these are reversed one to the other, so that at some intermediate latitude there will be no diurnal component. The semidiurnal oscillation apparent at intermediate latitudes has zero range at the equator and at the pole, and has a maximum value in the middle latitudes. Further, the north component has an apparently " constant term " for latitudes other than 0°. This term varies with latitude and it will be seen later that it varies with the lunar declination.

It is clear, however, that while the variations in tractive force are of apparent complexity, yet they can be split up into simpler parts. An example of this is given for the north component in Fig. 3.6, whose components are graphed in Fig. 3.9.

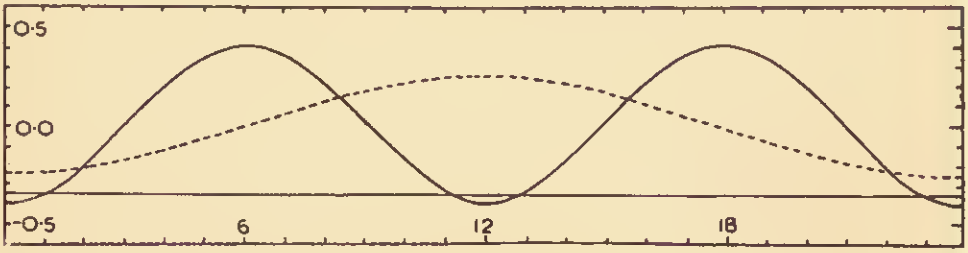

FIG. 3.9. Diurnal, semidiurnal and long-period components of north components of lunar tractive forces.
(Latitude 30° N., lunar declination 15° N.)

The method of construction of this graph is of some interest. The data utilised in constructing Fig. 3.9 are found in Table 3.6 N., and the mean of the values for hours 0 to 23 is readily found to be −0·35. Let this be subtracted from the entries for hours 0 to 23 and then arrange the results in two groups of 12 each, the first for hours 0 to 11 and the second for hours 12 to 23 as follows :—

Hours 0–11 :

−0·65 −0·59 −0·42 −0·17 0·08 0·29 0·41 0·42 0·33 0·18 0·02 −0·11

Hours 12–23

−0·15 −0·11 0·02 0·18 0·33 0·42 0·41 0·29 0·08 −0·17 −0·42 −0·59

Now the semidiurnal part will repeat itself after 12 hours while the diurnal part will repeat itself but with change of sign after 12 hours. Thus the semidiurnal part is obtained by adding the pairs of figures and dividing by 2, while the diurnal part is obtained by subtracting the lower figures from the upper figures and dividing by 2.

Hence we get the semidiurnal part as

Hours 0–11 :

−0·40 −0·35 −0·20 0·00 0·20 0·36 0·41 0·36 0·20 0·00 −0·20 −0·35

and the diurnal part as

Hours 0–11 :

−0·25 −0·24 −0·22 −0·18 −0·13 −0·07 0·00 0·07 0·13 0·18 0·22 0·24

The resolution of the tractive forces, therefore, into north and east components, and these again into their "constant," diurnal and semidiurnal parts, offers the simplest and most general method of considering them.

3.4. The effects of declination

It is instructive to prepare diagrams like that of Fig. 3.3 for various values of the lunar declination. It is not necessary to consider values in excess of 30°. The case of zero declination is important, and is illustrated in Fig. 3.10 for latitudes 30° and 60° N. It is apparent that the magnitudes of the forces at hours 0 and 12, 1 and 11, 2 and 10, are equal in pairs, and therefore the forces at hours 1 and 13, 2 and 14, are equal in pairs.

Hence we have the important result that **there is no diurnal inequality in the forces when the moon is on the equator.**

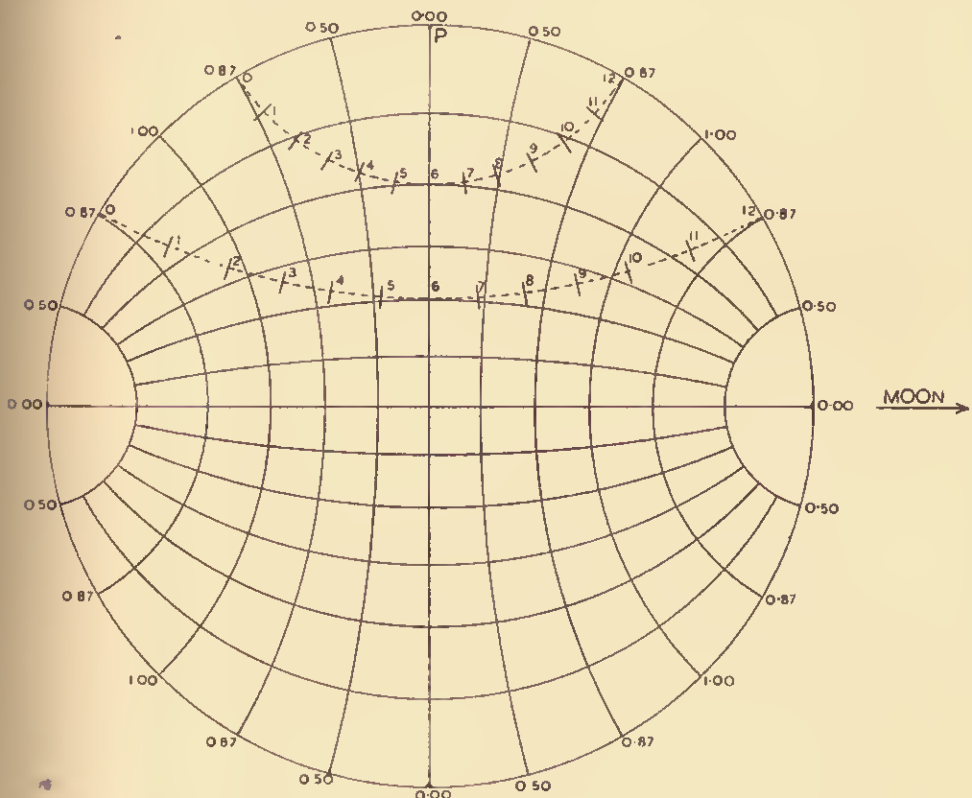

FIG. 3.10. Variations of lunar tractive forces in certain latitudes, hour by hour, with moon on equator. (P is the position of the north pole of the earth.)

An examination of the case where the lunar declination is 30° N. will show that the diurnal variations in tractive force are then nearly twice what they are for the case where the declination is only 15° N. It can be verified by the student that the semidiurnal variations are also susceptible to the influence of lunar declination, and that the amplitudes of the semidiurnal variations in the tractive forces decrease with increasing declination.

The deductions given above, though they can be obtained in the manner indicated, are summarised in mathematical formulæ as follows :—

Let d = the declination of the moon, considered as positive when the moon is north of the equator ;

l = the latitude of the place at which forces are required to be known ;

Z = the angular distance between the meridian of the place and the meridian at which it is then "lunar midnight," that is, at which lower transit is taking place.

Then the north component of the tractive force is proportional to

$$-\tfrac{1}{2}\sin 2l\,(1 - 3\sin^2 d) - \cos 2l \sin 2d \cos Z - \tfrac{1}{2}\sin 2l \cos^2 d \cos 2Z \quad . \quad (3.4a)$$

and the east component is proportional to

$$\sin l \sin 2d \sin Z - \cos l \cos^2 d \sin 2Z \quad . \quad . \quad . \quad . \quad (3.4b)$$

Both expressions need to be multiplied by the factor

$$\frac{3}{2} g \frac{M}{E} \frac{e^3}{r^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.4c)$$

in order to give true values of the components. It will be easily verified that these formulæ are in conformity with the conclusions already reached. For the proof of the formulæ see Art. 3.6.

3.5. General deductions concerning tides and tidal streams

The importance of a knowledge of the tide-generating forces was stressed at the beginning of Chapter II. Hitherto no assumptions have been made as to the response of the water to the forces, so that the knowledge we have acquired as to the variations of the forces is strictly accurate. It is now possible to assert that it is a simple matter of common-sense to deduce that somehow or other, in greater or less degree, with relatively unknown amplitudes, and at unknown times relative to the forces, the variations in the forces will naturally lead to similar variations in the actual tides.

Hence it is possible to make the following very important deductions :—

(1) Tides caused by the moon will recur at intervals of a lunar day.

(2) There will generally be two well-marked species of oscillations, one having a period of a lunar day and the other having a period of half a lunar day.

(3) Both species of oscillations will be affected in amplitude by the changing distance of the moon, and they will probably change approximately according to the cube of the lunar parallax, this effect being shown from the occurrence of the factor $(e/r)^3$ in the expression for the forces.

(4) The declination of the moon will play a very important part, the range of the semidiurnal tides being diminished at times of high declination, and the diurnal tide approximately being proportional to the declination of the moon.

(5) Tidal streams will be set in motion by the tractive force, and the characteristics of the streams will be similar to those of the forces. It will be necessary, for instance, to resolve the streams into their components and these again into the diurnal and semidiurnal parts in order to consider their variations.

(6) The solar forces will bring about variations in the tides and in the tidal streams in much the same way as the lunar forces, but because the solar forces are generally less than half the lunar forces in absolute magnitudes the solar tides will be somewhat less than half the lunar tides, on the average.

*3.6. Mathematical investigation of the variation of tractive force

It is impossible to examine accurately the variation of the tractive force without recourse to spherical trigonometry, but it will only be necessary to make use of two well-known formulæ connecting the sides (a, b, c) and angles (A, B, C) of a spherical triangle :—

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \quad . \quad . \quad . \quad . \quad (3.6a)$$

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \quad . \quad . \quad . \quad . \quad (3.6b)$$

* See par. 1, page vii.

Referring to Fig. 3.11, as in Art. 2.2 we shall denote by

O, O′ the centres of the earth and moon, respectively ;
U, U′ the points in which respectively O′O, and O′O produced, cut the earth's surface ;
X a point in the great circle in which any plane through OO′ cuts the surface of the earth ;
C the angle XOO′.

We shall also denote by

P the north pole of the earth ;
d the north declination of the moon ;
l the north latitude of the place X ;
Z the longitude of the place X, east of the meridian PU′.

In terms of lines and angles on the spherical surface depicted in Fig. 3.11 we thus have

$$\left.\begin{aligned} PX &= 90° - l \\ PU &= 90° - d \\ XU &= C \\ XPU &= 180° - Z \end{aligned}\right\} \quad \text{(3.6c)}$$

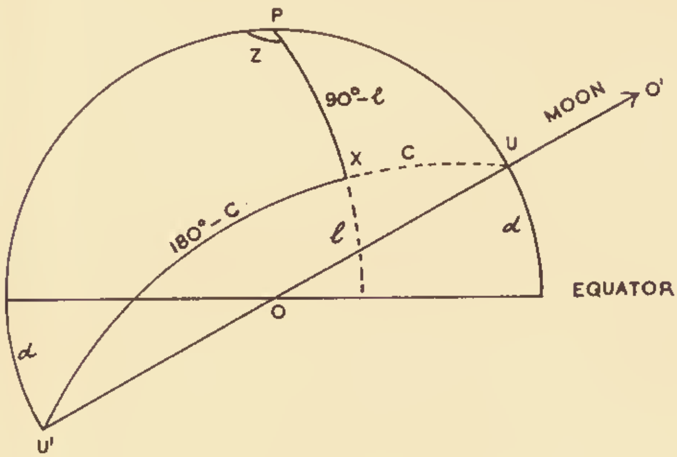

Fig. 3.11. Relation of angles d, l, C, Z.

Now the *tractive force* is proportional to

$$\sin 2C \quad (\text{or } 2 \sin C \cos C),$$

according to (2.6l), and the north and east components are therefore given respectively by

$$2 \sin C \cos C \cos PXU \quad \text{(3.6d)}$$

and

$$2 \sin C \cos C \sin PXU \quad \text{(3.6e)}$$

since the tractive force is directed along XU. These components have to be expressed in terms of the angles l, d and Z, and on considering the application of the formulæ (3.6a) and (3.6b) to the spherical triangle XPU we obtain

$$\cos C = \cos(90°-l)\cos(90°-d) + \sin(90°-l)\sin(90°-d)\cos(180°-Z) \quad \text{(3.6f)}$$

$$\cos(90°-d) = \cos(90°-l)\cos C + \sin(90°-l)\sin C \cos PXU \quad \text{(3.6g)}$$

$$\sin C \sin PXU = \sin(90°-d)\sin(180° - Z) \quad \text{(3.6h)}$$

These can be simplified to give

$$\cos C = \sin l \sin d - \cos l \cos d \cos Z \quad . \quad (3.6i)$$

$$\cos l \sin C \cos PXU = \sin d - \sin l \cos C \quad . \quad . \quad . \quad (3.6j)$$

$$\sin C \sin PXU = \cos d \sin Z \quad . \quad . \quad . \quad . \quad (3.6k)$$

On substituting by (3.6j) in (3.6d) we get

$$2 \cos C\ (\sin d - \sin l \cos C)/\cos l$$

and on substitution for cos C by (3.6i) we get

$$2\ (\sin l \sin d - \cos l \cos d \cos Z)\ (\sin d - \sin^2 l \sin d + \sin l \cos l \cos d \cos Z)/\cos l$$

$$= 2\ (\sin l \sin d - \cos l \cos d \cos Z)\ (\cos l \sin d + \sin l \cos d \cos Z)$$

$$= 2 \sin l \cos l \sin^2 d - 2 \cos Z\ (\cos^2 l \sin d \cos d - \sin^2 l \sin d \cos d) - 2 \cos^2 Z\ (\sin l \cos l \cos^2 d)$$

and, finally, on writing

$$2 \cos^2 Z = 1 + \cos 2Z$$

we get the **north component of tractive force** is proportional to

$$-\tfrac{1}{2} \sin 2l\ (1 - 3 \sin^2 d) - \cos 2l \sin 2d \cos Z - \tfrac{1}{2} \sin 2l \cos^2 d \cos 2Z \quad . \quad (3.6l)$$

By substituting from (3.6i) and (3.6k) in (3.6e) we get the **east component of tractive force** is proportional to

$$\sin l \sin 2d \sin Z - \cos l \cos^2 d \sin 2Z \quad . \quad . \quad . \quad (3.6m)$$

The actual values of tractive force are to be found by multiplying the expressions (3.6l) and (3.6m) by the general factor

$$\frac{3}{2} g \frac{M}{E} \frac{e^3}{r^3} \quad . \quad . \quad . \quad . \quad . \quad (\text{see } 2.6b)$$

Since the angle Z increases by 15° per lunar hour then the values of the components of tractive force can be computed for any given values of l, d, or r by taking Z at intervals of 15° and the results are then given at lunar hours. The values given in the following tables are for the case $d = 15°$, $l = 0°$, 30°, 60° and 89°·9, the general factor being omitted.

North component, $d = 15°$

$$\left.\begin{aligned} l &= 0° &&: && - 0{\cdot}500 \cos Z \\ l &= 30° &&: - 0{\cdot}346 && - 0{\cdot}250 \cos Z - 0{\cdot}404 \cos 2Z \\ l &= 60° &&: - 0{\cdot}346 && + 0{\cdot}250 \cos Z - 0{\cdot}404 \cos 2Z \\ l &= 89°{\cdot}9 &&: - 0{\cdot}001 && + 0{\cdot}500 \cos Z - 0{\cdot}001 \cos 2Z \end{aligned}\right\} \quad . \quad . \quad (3.6n)$$

East component, $d = 15°$

$$\left.\begin{aligned} l &= 0° &&: && - 0{\cdot}933 \sin 2Z \\ l &= 30° &&: 0{\cdot}250 \sin Z && - 0{\cdot}808 \sin 2Z \\ l &= 60° &&: 0{\cdot}433 \sin Z && - 0{\cdot}467 \sin 2Z \\ l &= 89°{\cdot}9 &&: 0{\cdot}500 \sin Z && - 0{\cdot}003 \sin 2Z \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3.6o)$$

The above formulæ form the basis of Tables 3.5 to 3.8.

3.7. Notes on the construction of stereographic diagrams

The diagrams used in this chapter are extremely valuable for giving an insight into the variations of the tractive forces, and it may therefore be useful to indicate how students and teachers may construct such charts. Draw a circle with centre O and radius a and take two diameters QOQ′ and ROR′ at right angles, as in Fig. 3.12. Divide the circle into equal angles, say at intervals of 15° from the terminal points of the diameters, as in the examples in this chapter. Let P denote one of these

divisions of the circumference and join PR′ and PQ′, cutting the diameters at X and Y. Then we get the intercepts

$$OX = a \tan \tfrac{1}{2}(90° - \theta) \quad . \quad . \quad . \quad . \quad (3.7a)$$

$$OY = a \tan \tfrac{1}{2}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.7b)$$

A circle with centre T on RR′ and drawn through P and Y will denote a line of latitude. Its radius is

$$TY = a \cot \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.7c)$$

The circle with centre S on QQ′ and passing through R and X will denote a line of longitude. Its radius is

$$SX = a \sec \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.7d)$$

In drawing the figures, it is only necessary to compute the intercepts, radii and distances of centres as in the table below for angles at intervals of 15°, with a radius $a = 10$.

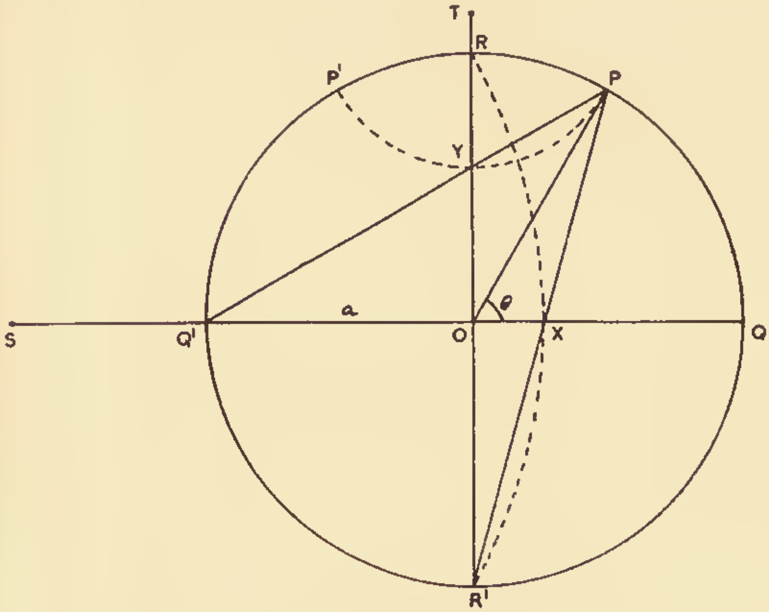

FIG. 3.12. Construction of stereographic projections.

RADIUS, OQ = 10

θ	Intercept OX	Intercept OY	Radius SX	Radius TY	OS	OT
15°	7·67	1·32	10·35	37·32	2·68	38·64
30°	5·77	2·68	11·55	17·32	5·77	20·00
45°	4·14	4·14	14·14	10·00	10·00	14·14
60°	2·68	5·77	20·00	5·77	17·32	11·55
75°	1·32	7·67	38·64	2·68	37·32	10·35

CHAPTER IV

THE EQUILIBRIUM TIDE

4.1. The equilibrium tide

In the tidal theory so far presented we have refrained from making any assumptions which will tend to create any sense of artificiality. It has been indicated, however, that the variations of the forces will probably yield similar variations in the tides, whether they are considered as existing upon the earth, or as existing under idealised conditions. Since it is highly desirable to have available a standard of reference so that tides as they exist can be compared with, and expressed in terms of, the standard, it has been customary to use for this purpose the equilibrium tide, as it is called. Unfortunately, in the past, much confusion has been caused by laying undue stress upon the equilibrium tide, and an impression has been given that it is

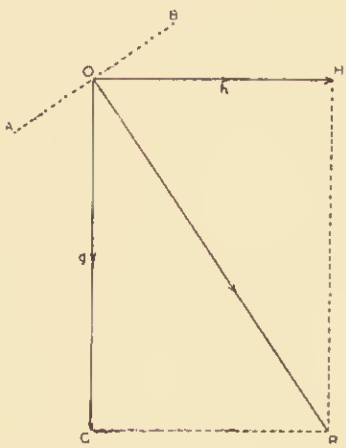

FIG. 4.1. Equilibrium surface under gravitational and lunar tractive forces.

only because of the perversity of nature that the equilibrium tide does not exist. In attempting to force existing tides into the mould of the equilibrium tide, so to speak, confusion has been caused and progress has been retarded.

The simple equilibrium theory postulates the absence of all land masses, and assumes that while the water retains its gravitational properties it has a negligible inertia so that it can respond instantaneously to the forces. Alternatively, the name of the theory indicates that an indefinite time is allowed for the waters to reach a state of equilibrium.

The equilibrium conditions will result when there is no resultant attractive force which will cause a particle to move. Now the only forces we need to consider are the vertical force of the earth's gravitation, and the tractive forces. These are given diagrammatically in Fig. 4.1 though the magnitude of the actual horizontal force is only a very, very small fraction of the force of gravity. Let OG represent the force of gravity (g) and OH the tractive force (h); then OR represents the resultant force, as was explained in Art. 2.1, and the surface of the fluid will be at rest when it is at right angles to the resultant force; *i.e.* in the figure the surface of the fluid is represented by AB.

It is clear that the angle HOB is equal to the angle GOR, and therefore the slope of the surface to the original horizontal direction is given by the ratio h/g.

Fig. 3.2 gives the values of the ratio h/g (apart from a general factor) at various points on the earth's surface.

Starting at V, the fluid surface will have zero gradient, and the gradient will slowly increase until the slope of the surface is greatest half-way between V and U, and thereafter the slope diminishes until the gradient is zero again in the line of centres at U. It is obvious that the surface will therefore take a form such that the water is drawn up above the surface of the earth at U (towards the moon on one side and away from it on the opposite side) and is depressed at the pole. The actual amounts of elevation, and the true shape, depend upon the necessity of ensuring that the whole volume remains unchanged, and this condition requires mathematical

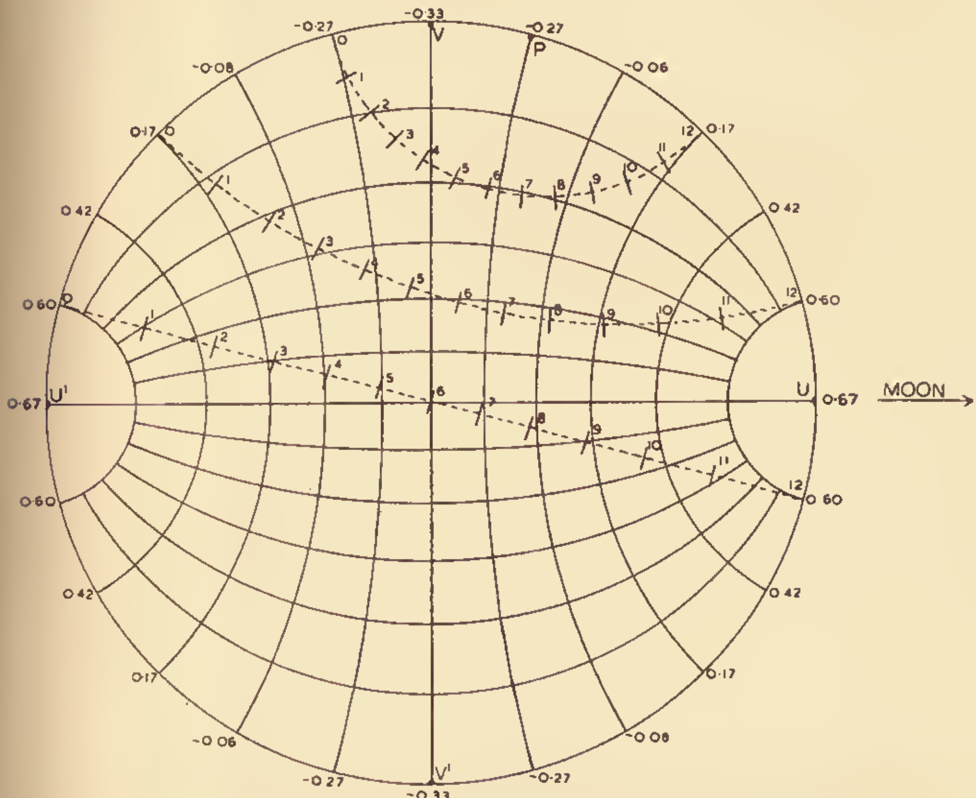

Fig. 4.2. Deduction of variations of lunar equilibrium tide in certain latitudes, hour by hour, with moon 15° N. of the equator. (P is the position of the north pole of the earth.)

treatment, as in Art. 4.5, but the actual values of the equilibrium tide (with a general factor of

$$\frac{3}{2}\frac{M}{E}\frac{e^3}{r^3}e$$

as obtained in Art. 4.5) are given in Fig. 4.2, which has been constructed on the same principles as in Art. 3.2, though the problem is simpler for only elevations are required. These values can be tested in a general way. Thus, the gradient is zero at V and V′, and on the two points U, U′ on the line of centres; also the maximum gradients are at points about half-way between the points of zero gradient.

4.2. The variations of the lunar equilibrium tide

The variations of the lunar equilibrium tide for the cases of latitudes 0°, 30°, 60° N. with lunar declination 15° N. as obtained from Fig. 4.2 are given in Fig. 4.3,

and it is at once obvious that the variations are composed of

(*a*) " constant " terms ;
(*b*) diurnal oscillations ;
(*c*) semidiurnal oscillations.

Many deductions can be made concerning the way in which the amplitudes vary. At the pole there is no variation of elevation (though the forces have diurnal variations there, it may be noted), and the diurnal tide is absent at the equator though quite evident elsewhere. The semidiurnal tide has its maximum amplitude at the equator. Such variations as these, are, of course, of only passing interest. The main deductions that can be readily made from Fig. 4.2 and similar figures are as follows :—

(*a*) The equilibrium tide is composed of "constant" terms, diurnal oscillations and semidiurnal oscillations.

(*b*) The diurnal part increases with the declination and vanishes with zero declination.

(*c*) The semidiurnal part decreases with declination.

(*d*) The tide varies with the lunar distance, approximately as the cube of the parallax.

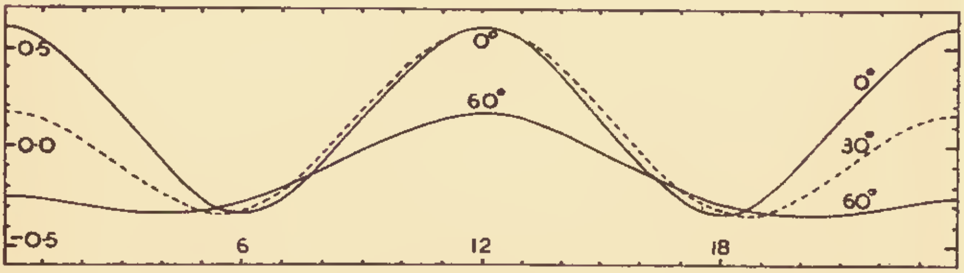

FIG. 4.3. Variations of lunar equilibrium tide in certain latitudes, hour by hour, with moon 15° N. of the equator.

It will be noted that similar deductions were made for tides in general from a consideration of the forces alone, as a matter of simple common sense. We have now shown that in a particular case of tidal motion, even though there is a large element of artificiality about it, the motions do show the variations indicated in a direct manner by the forces. Therefore we can the more confidently assert that the characteristic variations of the forces indicate the characteristic variations of the existing tides. It is now convenient to say, because the equilibrium tide and existing tides are alike in this respect, that both exhibit the variations of the forces, therefore

the characteristic variations of the equilibrium tide exhibit the characteristic variations of the existing tide, in greater or less degree.

It is on this basis, and on this alone, that we are able to justify the use that is made of the equilibrium tide as a standard of reference.

One important point may be noted, and that is that the effects of declination and parallax tend to appear as common factors to all tides of the same species ; that is, the diurnal forces and the diurnal equilibrium tide have factors depending upon parallax and declination which are quite independent of position upon the earth's surface. Similarly, if declination or parallax reduces the semidiurnal tide at any one place, it will reduce the semidiurnal tide at any other place in the same ratio. While it does not follow that existing tides will be governed exactly by this rule there is a strong presumption that the rule will be approximately followed.

The " constant " terms which have been referred to are constant only while the declination and the parallax are constant. There will be slight declinational or parallactic variations which will have periods of about a fortnight and a month, and these are called " long-period tides." The greater part of the " constant," however, is truly a constant, so that there is a definite average deformation of the earth's fluid surface due to this. The deformation is positive (*i.e* the surface is raised) over the equator, and negative (*i.e.*, the surface is permanently lowered) at the poles.

4.3. Formulæ for the equilibrium tide

It is shown in Art. 4.5 that the lunar equilibrium tide can be expressed by the following formulæ ; similar formulæ can be written for the solar equilibrium tide by replacing M by S and adding dashes to the symbols r, c, d, Z :—

$$\text{Common coefficient} \quad : \quad \frac{3}{2}\frac{\mathrm{M}}{\mathrm{E}}\frac{e^3}{c^3}e \quad . \quad . \quad . \quad . \quad . \quad (4.3a)$$

$$\text{Long-period species} \quad : \quad \frac{3}{4}\left(\frac{c}{r}\right)^3\left(\frac{1}{3}-\sin^2 l\right)\left(\frac{2}{3}-2\sin^2 d\right) \quad . \quad (4.3b)$$

$$\text{Diurnal species} \quad : \quad -\frac{1}{2}\left(\frac{c}{r}\right)^3 \sin 2l \sin 2d \cos \mathrm{Z} \quad . \quad . \quad (4.3c)$$

$$\text{Semidiurnal species} \quad : \quad \frac{1}{2}\left(\frac{c}{r}\right)^3 \cos^2 l \cos^2 d \cos 2\mathrm{Z} \quad . \quad . \quad (4.3d)$$

where

e = the radius of the earth ;
r = the distance of the moon ;
c = the mean value of r ;
d = the declination of the moon, considered positive with north declination ;
l = the latitude of a place on the earth, considered positive when north ;
Z = the angular distance between the meridian of the place, and the meridian at which lower transit is taking place ;

(with similar definitions for the dashed variables r', c', d', Z' used for the solar formulæ).

The mean value of the common coefficient, with

$$\left.\begin{aligned} \mathrm{M/E} &= 1/81{\cdot}53 \\ e/c &= 1/60{\cdot}26 \\ e &= 20{,}900{,}000 \text{ ft.} \end{aligned}\right\} \quad . \quad . \quad . \quad (4.3e)$$

is 1·76 ft.

Hence, with average value of parallax, the lunar equilibrium semidiurnal tide has a maximum amplitude of 0·88 ft. in latitude 0°, when the declination is zero, and the maximum amplitude of the lunar equilibrium diurnal tide is also 0·88 ft., occurring in latitude 45° when the declination is 45°.

The analysis for the solar equilibrium tide is similar to that for the lunar tide, and the maximum amplitudes given above for the lunar tide are reduced in the ratio 0·460 (as obtained in Art. 2.4), thus becoming 0·40 ft.

4.4. The necessity for further analysis

We have pointed out that the equilibrium tide manifests the characteristics of the tide-generating forces, and hence we should expect similar variations in actual tides.

Now the factor c^3/r^3 is common to all species so that as all the forces everywhere are affected by this parallax factor in the same ratio ; therefore the lunar tides, we might anticipate, would be affected in the same ratio.

Further, the diurnal forces, for instance, have a common factor depending upon declination, so that it might reasonably be anticipated that the lunar diurnal tides all over the earth would be simultaneously affected to the same relative degree by

declinational changes, large and small diurnal tides being increased or reduced in the same ratio. The declinational variations of the semidiurnal tides would also appear to be much the same everywhere, but as the effects of declination are different with the three species it would be absolutely necessary to deal separately with the different species. Similarly the solar tides would need separate treatment from the lunar tides, as the parallax and declination of the sun differ from those of the moon.

Hence, even if long-period tides can be ignored for approximate tidal predictions it is absolutely essential to consider separately the variations of four essential contributions:

(*a*) the lunar semidiurnal tide ;
(*b*) the solar semidiurnal tide ;
(*c*) the lunar diurnal tide ;
(*d*) the solar diurnal tide.

The time of high water of the lunar equilibrium semidiurnal tide occurs when $\cos 2Z = 1$; that is when $Z = 0°$ or $180°$, or at the local times of lower transit and upper transit. For a place in the northern hemisphere, the lunar equilibrium diurnal high water occurs at the time of upper transit with the moon in north declination ($\sin 2d$ positive, $\cos Z$ negative), and at the time of lower transit with the moon in south declination ($\sin 2d$ negative, $\cos Z$ positive). In nature, this cannot be expected to occur, but a principle, which in effect we have already used, was formulated by Laplace, asserting that if the forces vary with a known periodicity then the tides will exhibit the same periodicity. Hence this principle would appear to justify the assumption that tbe times of high water of the lunar semidiurnal tides would follow the times of transit by a time lag which would vary with geographical position. Similarly, the lunar diurnal high water at a place would follow the time of upper transit of the moon by a definite constant time appropriate to the place. These two *intervals*, of course, need not be the same for both species.

It will be noticed that this principle is essentially independent of any theory of tidal motion upon the earth, but it can only be strictly true when the periodicity of the forces is absolutely regular.

It was found by investigators of tidal phenomena that the relations indicated above were approximately true for tides upon the earth, but great difficulty was experienced in dealing with the four entities—the lunar and solar diurnal and semidiurnal tides—so as to give the whole tide. Moreover, it was found that the separate entities could not always be treated for parallactic and declinational changes in the simple manner indicated by the forces, and the reason for this will now be indicated.

It is a matter of common experience that every body of water will oscillate with its own natural period of oscillation, whether it be water in a dish or in a pond or in an ocean. Unless maintained by some external force the oscillation will diminish because of frictional forces and will ultimately die away. A swing is maintained in oscillation by a child when he gives a little impulse to its motion always at the same position of the swing ; that is, in phase with the swing. Erratic impulses will not maintain the motion, but if the impulses are nearly in phase they will maintain the oscillation for a much longer period than would otherwise occur. When the swing is always moving through the same arc then the impulses given to it are just sufficient to counteract the decay due to friction.

Now the tidal forces have periods of about 12 hours for the semidiurnal species and 24 hours for the diurnal species. The period of oscillation of a body of water depends upon the depth, and also upon the surface dimensions, and in a simple basin of depth h ft. and length l ft. the period varies as $l/\sqrt{gh}$, where g is the constant of gravity. (This will be proved in Chapter XVIII.) In general, therefore, an ocean would need to be very large and of no great depth, for the period of natural vibration to be 24 hours. The chances of a natural vibration of 12 hours are much greater, for the surface need not be so great, nor the depth so small, as would be needed for a free period of 24 hours. For the reason that because the physical dimensions of the earth are such as to favour free periods of the order of 12 hours rather than periods of the order of 24 hours, it follows that semidiurnal tides tend to be greater than

diurnal tides—in other words, the response to the forces is often greater for semidiurnal tides than it is for diurnal tides.

This variation of response with periodicity of forces is, of course, seen more clearly in the cases of different species, but the effects of the natural configurations are to be seen in each species. Instead of the actual semidiurnal tide always being related to the equilibrium tide by a constant factor and a constant time-lag, it is found that the factor and the time-lag vary somewhat with the length of the lunar day. Laplace's principle would teach that the tide would show the periodicity of the forces, but if the lunar day varies in length then the periodicity of the forces changes also, and consequently the response to the forces. Similarly, the diurnal tide is related to the equilibrium diurnal tide by a factor and time-lag which varies a little with the length of the lunar day.

It is fairly common knowledge, and it can be verified from any table of times of transit of the moon, that the lunar day varies, and therefore we have to examine the degree of this variation, and generally to consider in greater detail the lunar and solar motions.

*4.5. Mathematical investigation of the equilibrium tide

Referring to Art. 2.4 and Fig. 3.2, the ratio of the horizontal component of the lunar tide-generating force to the force of gravity is given by the product of

$$\frac{3}{2}\frac{M}{E}\frac{e^3}{r^3} \text{ and } \sin 2C \qquad . \quad . \quad . \quad . \quad (4.5a)$$

It must be noted, however, that the corresponding surface gradient will be negative in the direction of increasing C, and that temporarily we can ignore the first factor as it does not vary with the position of a point upon the earth. It is a simple matter to deduce that the elevation corresponding to the surface gradient of

$$-\sin 2C$$

in the direction of increasing C must be given by such an expression as

$$A + B\cos 2C$$

where A and B are constants to be determined.

Take a small arc between radii at angles $(C + \alpha)$ and $(C - \alpha)$ whence the intercepted arc of the earth's surface is 2α multiplied by the radius (e) of the earth. Hence the gradient of $B\cos 2C$ is given by

$$B\frac{\cos 2(C+\alpha) - \cos 2\,(C-\alpha)}{2\alpha e}$$

which is equal to

$$-\frac{B}{e}\frac{2\sin 2C\sin 2\alpha}{2\alpha} \qquad . \quad . \quad . \quad . \quad . \quad (4.5b)$$

When α is very small we have $\sin 2\alpha$ equal to 2α, so that the gradient is $-\sin 2C$ if $B = \frac{1}{2}e$.

To determine the value of A we must consider the total volume of displaced water. Now the elevation depends upon the angle C for all planes through the centres of the moon and earth, so that, as in Fig. 3.2, the elevation is constant along one of the parallel circles. Consider two such circles very close together, separated by only a small angle α and let C be the angle between the middle of the strip, and the line of centres of the moon and earth. Then the radius of the centre of the strip is $e\sin C$, its width is αe, the elevation upon it has a mean value of

$$A + \tfrac{1}{2}e\cos 2C$$

so that the total volume of displaced water on the strip is

$$2\pi e\sin C\,(A + \tfrac{1}{2}e\cos 2C)\,\alpha e \qquad . \quad . \quad . \quad . \quad (4.5c)$$

* See par. 1, page vii.

The total volume of displaced water on all such small strips must be zero in order neither to gain nor lose on the original volume of the undisturbed water. Hence the average value of

$$\sin C\,(A + \tfrac{1}{2}e \cos 2C) \qquad . \quad . \quad . \quad . \quad (4.5d)$$

must be zero ; that is, the average value of

$$A \sin C - \tfrac{1}{4}e \sin C + \tfrac{1}{4}e \sin 3C \quad . \quad . \quad . \quad . \quad (4.5e)$$

must be zero, for all values of C from 0 to π.

Now it is obvious that if C goes through π, then the values of 3C go through 3π, and that the values between π and 2π will cancel out the values between 2π and 3π, so that the average value of sin 3C is one-third of the average value of sin C. Hence for the average value of the above expression to be zero we must have

$$A - \tfrac{1}{4}e + \tfrac{1}{12}e = 0, \ \text{so that } A = \tfrac{1}{6}e \quad . \quad . \quad . \quad (4.5f)$$

Hence the lunar equilibrium tide is given by

$$\frac{3}{2}\frac{M}{E}\frac{e^3}{r^3}e\,(\tfrac{1}{6} + \tfrac{1}{2}\cos 2C) \quad . \quad . \quad . \quad . \quad (4.5g)$$

which is used to give the data of Fig. 4.2.

The reduction of this expression to one in terms of latitude and declination follows the same lines as the investigation of forces in Art. 3.6, except that it is simpler.

We can write successively, using (3.6i),

$$\tfrac{1}{6} + \tfrac{1}{2}\cos 2C = \cos^2 C - \tfrac{1}{3} = (\sin l \sin d - \cos l \cos d \cos Z)^2 - \tfrac{1}{3}$$

$$= (\sin^2 l \sin^2 d + \tfrac{1}{2}\cos^2 l \cos^2 d - \tfrac{1}{3}) - \tfrac{1}{2}\sin 2l \sin 2d \cos Z + \tfrac{1}{2}\cos^2 l \cos^2 d \cos 2Z$$

$$= \tfrac{3}{2}(\sin^2 l - \tfrac{1}{3})(\sin^2 d - \tfrac{1}{3}) - \tfrac{1}{2}\sin 2l \sin 2d \cos Z + \tfrac{1}{2}\cos^2 l \cos^2 d \cos 2Z \quad . \quad (4.5h)$$

The above expression, multiplied by the factor

$$\frac{3}{2}\frac{M}{E}\frac{e^3}{r^3}e$$

gives the lunar equilibrium tide.

It will be noted that three species of tides exist, long-period tides due to the first term, diurnal tides due to the second term, and semidiurnal tides due to the third term, since the angle Z changes by about 360° per day.

The expression (4.5h) is used to give the data of Fig. 4.3, the variations of tide being proportional to the following expressions :—

Declination = 15° N.

$$\left.\begin{array}{lll} \text{Latitude } 0^\circ & : & 0{\cdot}133 \qquad\qquad\qquad\quad + 0{\cdot}467 \cos 2Z \\ \text{Latitude } 30^\circ \text{ N.} & : & 0{\cdot}033 - 0{\cdot}217 \cos Z + 0{\cdot}349 \cos 2Z \\ \text{Latitude } 60^\circ \text{ N.} & : & -0{\cdot}167 - 0{\cdot}217 \cos Z + 0{\cdot}117 \cos 2Z \end{array}\right\} \quad (4.5i)$$

The solar equilibrium tide can be readily expressed in a similar manner.

CHAPTER V

MOTIONS OF THE SUN AND MOON

5.1. The motions of the sun and moon

Beginning with the motion of the sun, it is well known that the apparent path of the sun through the heaven is in a plane cutting what is known as the celestial sphere in a path called the ecliptic. The plane of the ecliptic is inclined at a constant angle of 23° 27′ to the plane of the equator. The point where the apparent path of the sun crosses the equator from south to north is called, by historical custom, "the first point of Aries" (denoted by ♈), or, alternatively, "the vernal equinox" (see Fig. 5.1).

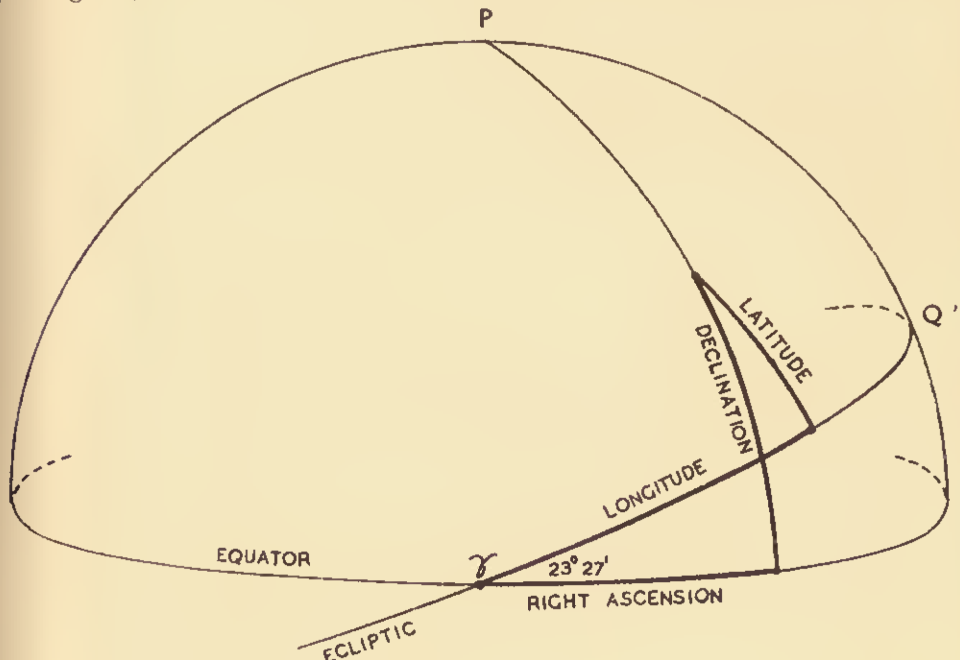

Fig. 5.1. Astronomical angular definitions.

The position of any heavenly body is defined by (1) its "longitude," which is the angular distance eastward along the ecliptic, measured from the vernal equinox, and (2) by its "latitude," measured positively to the north of the ecliptic along a great circle cutting the ecliptic at right angles. But, alternatively, its position can be expressed in terms of its angular distance along the equator from the vernal equinox (this angle being called the "right ascension"), together with its angular distance (called the "declination"), north or south of the equator, measured along a meridian.

The apparent motion of the sun is such that a complete revolution of the ecliptic is made in a mean solar year of 365·2422 mean solar days, and its movement in latitude is negligible.

The apparent path of the moon oscillates somewhat about the ecliptic (Fig. 5.2), and observation shows that while the moon completes a revolution, measured along the ecliptic, in a period of 27·3216 mean solar days, the cycle of oscillation north

and south of the ecliptic is completed in 27·2122 mean solar days. Thus the moon returns approximately to the same point in a little over a month of 27 days, but the difference between the period of revolution in the orbit and the period of oscillation north and south of the equator is of considerable importance. Suppose that the moon commences a cycle at the vernal equinox. When the moon has completed its cycle north and south of the ecliptic, the revolution in orbit is not complete by 0·1094 day, on the average. The "ascending node" (that is, the point where the moon crosses from south to north of the ecliptic), thus travels westwards (*i.e.*, backwards, since the moon and sun travel eastwards), by 0·1094 day in every 27·2122

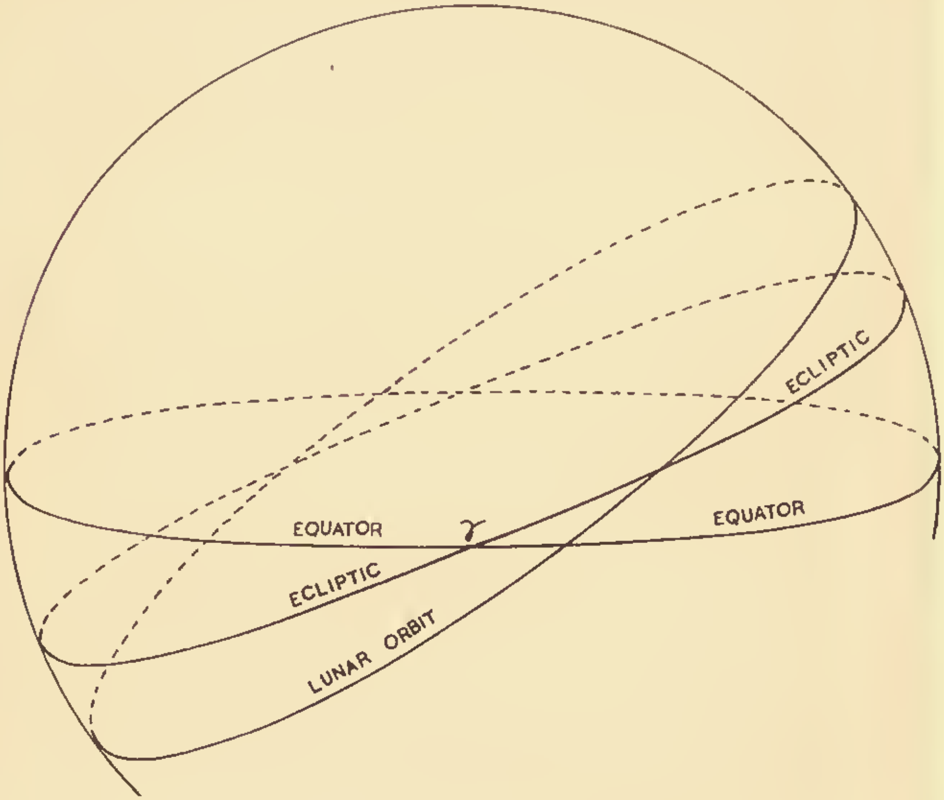

Fig. 5.2. Relations between equator, ecliptic and lunar orbit.

days, and therefore on the average this "regression of the nodes" will be completed in

$$\frac{27{\cdot}3216}{0{\cdot}1094} \text{ cycles in latitude}$$

$$\text{or in } \frac{27{\cdot}3216}{0{\cdot}1094} \times \frac{27{\cdot}2122}{365{\cdot}25} \text{ years} = 18{\cdot}61 \text{ Julian years.}$$

A Julian year is equal to 365·25 mean solar days.

Further, it is known from observations that the lunar orbit is in a plane which is inclined to the plane of the ecliptic by practically a constant angle of 5° 8′, and the variations from this figure are negligible, so far as we are concerned. It is clear from Fig. 5.2 that the maximum declination of the moon will occur when the ascending node is at the vernal equinox (♈), when the north declination will rise during the following month to about 5° above the ecliptic (23° 27′ + 5° 8′ = 28° 35′ in all) and a

fortnight later the south declination will have an equal value. If, however, the descending node is at the vernal equinox, the maximum declination (north or south) will not be more than 23° 27′—5° 8′ = 18° 19′. These values will recur at intervals of 18·61 years so that we have the important fact tbat any tidal variations associated with lunar declination will have a regular variation in a period of 18·61 years (generally referred to briefly as the nineteen-yearly variation).

The phenomena associated with the nineteen-yearly period are so interesting and so important that it may be well to consider the matter a little more generally. For this purpose we shall refer to the periods of recurrence of lunar phases and of lunar distance, which are obtained from observations, so that in all we have four "months" as follows :—

(*a*) 29·5306 mean solar days for the period of recurrence of lunar phases, this being the period generally referred to as a lunation ;
(*b*) 27·5546 mean solar days for the period of oscillation in lunar distance ;
(*c*) 27·2122 mean solar days for the period of oscillation of the moon in latitude ;
(*d*) 27·3216 mean solar days for the period of revolution of the moon in longitude.

From the last two we have already deduced the period of revolution of the moon's nodes, and it is clear that the moon's motions will tend to recur after this period in the sense that the moon will cross the ecliptic at the same point after 18·61 years. But the sun will not be in the same position relatively to the moon after that period, for the sun is only found in the same position in the ecliptic after an exact mean solar year. It is a very remarkable fact, discovered long ago by Meton, that 235 lunations occur almost exactly in 19 mean solar years. Again, it has been known for many centuries that eclipses tend to recur in a period of nearly 18 years and 11 days. Now the extent of an eclipse is largely governed by the distance of the moon as well as its nearness to the ecliptic, so that this period is related to the months (*a*) and (*b*), in that it includes 223 lunations and 239 oscillations in lunar distance, almost exactly.

Hence we get three important periods :—

(1) 18·61 years as the period of revolution of the moon's nodes ;
(2) 19·00 years, the Metonic cycle, giving the recurrence of lunar phases
(3) 18·03 years, the Saros, giving the recurrence of eclipses.

The question is often asked as to whether these periods have any direct use in avoiding the necessity of predicting tides. The answer is that if the Saros is used the lunar declination is not repeated exactly, and that the extra 11 days on the exact number of years affects the solar tides, while if the Metonic cycle is used the lunar distances are not the same. The necessary corrections to the observations of 19 years ago would be as troublesome as the direct prediction, and of course the meteorological influences on the old observations would not be easily corrected.

5.2. Formulæ for solar motions

It is impossible to enter into the details of solar and lunar theory, and so we accept the formulæ given by astronomers as representing the solar and lunar motions. These formulæ are indicated by theoretical investigations of the motions of bodies under known attractive forces, and it has been found that the theoretical results agree extremely well with the observations. The degree of accuracy is far beyond what is required in tidal theory, so that for our purposes we can accept the computed values of latitude or longitude, declination or right ascension, in the Nautical Almanac as being in every way equivalent to direct observations. Hence we can accept the mathematical formulæ and satisfy ourselves if necessary by comparison with the values given in the Nautical Almanac. Most of the formulæ have been restricted to the principal terms, so that they are only approximations, but they are sufficiently accurate for our purposes.

The most usual method of stating the positions of the sun and moon is by referring their movements to the ecliptic, but as we have obviously important reasons

for desiring to use formulæ for declination we shall therefore give certain formulæ connecting these with latitude and longitude.

Further, the actual values of longitudes, being variable quantities increasing at variable rates, are usually expressed in terms of " mean longitudes," which increase at steady rates. As an example of this the mean sun may be considered. This is a fictitious body supposed to move steadily round the ecliptic at a rate which is the average rate of the true sun, and its longitude is said to be the mean longitude of the sun. The idea is extended to include a " mean moon," and also to mean longitudes of special positions in orbits, such as the perigee, the point of nearest approach to the earth.

Let r' = the distance of the sun ;
c' = its mean distance ;
x' = the longitude of the sun ;
h = the mean longitude of the sun, or the longitude of the " mean sun," increasing by $0^\circ{\cdot}0411$ per mean solar hour ;
p' = the mean longitude of the perigee, this angle being practically a constant angle ;
a' = the right ascension of the sun ;
d' = the declination of the sun.

One of the consequences of the law of gravitation is that the apparent angular rate of movement of a body in its orbit falls off as the square of its distance from the attracting body. Consequently, for the earth in relation to the sun, we are informed that :

$$c'/r' = 1 + 0{\cdot}017 \cos (h - p') \qquad (5.2a)$$

$$x' = h + 0{\cdot}034 \sin (h - p') \qquad (5.2b)$$

The first formula simply expresses the fact that the distance of the sun changes by as much as the mean distance multiplied by 0·017, which is an observational fact, and the cosine of $(h - p')$ indicates that the value of c'/r' is greatest when the sun is at perigee, which is, of course, obvious by definition. The second formula indicates that the rate of change in x' for a small change in time (*i.e.*, in h also) is greatest when $h = p'$, for if we were to graph the expression we should see that the sine term is changing most rapidly when $(h - p')$ is 0° or 180°. In the former case the rate of change of x' is greater than that of h and in the latter case it is smaller ; that is, the greatest rate of change takes place when the sun is at its nearest point to the earth, and the least rate of change when the sun is farthest from the earth. The coefficient 0·034 is twice the coefficient 0·017, and this is a consequence of the fact already pointed out, that the rate of change varies with the square of the distance.

We also have a simple relation between the declination and the longitude, given by

$$\sin d' = \sin (23^\circ\, 27') \sin x' = 0{\cdot}398 \sin x' \qquad (5.2c)$$

We readily see from Fig. 5.1 that the declination is greatest when x' is 90°, and the maximum declination is 23° 27′, the well-known angle associated with the tropics of Cancer and Capricorn and the Arctic and Antarctic circles. Also d' and x' are zero together, which occurs when the sun is at ♈. Hence we can accept this formula as sufficiently accurate for our present purposes.

It is not very easy to connect together the right ascension and the longitude, but clearly both are zero together when the declination is zero (the sun at ♈), and both are 90° when the sun is at maximum declination (the point Q in Fig. 5·1). But for values of x' less than 90° the right ascension must be less than the longitude ;

for instance, consider a point near ♈ ; then the longitude is the longest side of the right-angled triangle formed by the equator, the ecliptic and the meridian through the sun. This difference must reach a maximum and then decrease to zero when $x' = 90°$. Hence we see the reasonableness of the formula

$$a' = x' - 0{\cdot}043 \sin 2x' \quad . \quad . \quad . \quad . \quad (5.2d)$$

The effect of this is that the rate of movement in the equator is least when the sun is at ♈ ; that is, the rate of movement in the equator is least with zero declination.

The relation between the declination and the right ascension is given by :

$$\sin d' = 0{\cdot}406 \sin a' + 0{\cdot}008 \sin 3a' \quad . \quad . \quad . \quad . \quad (5.2e)$$

which can be verified for actual cases such as $x' = 0°, 45°, 90°$, and which gives results in agreement with the formulæ already quoted for $\sin d'$ and a' in terms of x'.

The method of deduction of such a formula, however, is of interest, as the harmonic development is based upon similar principles. In the mathematical work all angles will be understood as being given in radians, and much use will be made of the fact that when an angle is small then the sine of the angle can be taken as being equal to the angle itself, in radian measure, and the cosine of a small angle can be taken as unity.

We have

$$\sin d' = 0{\cdot}398 \sin (a' + 0{\cdot}043 \sin 2x')$$

and the angle $0{\cdot}043 \sin 2x'$ is so small and the angle x' is so nearly equal to the angle a' that we can replace x' by a', and proceed to obtain

$$\begin{aligned}\sin d' &= 0{\cdot}398 \sin a' \cos (0{\cdot}043 \sin 2a') + 0{\cdot}398 \cos a' \sin (0{\cdot}043 \sin 2a') \\ &= 0{\cdot}398 \sin a' + 0{\cdot}017 \cos a' \sin 2a' \\ &= 0{\cdot}398 \sin a' + 0{\cdot}008 (\sin 3a' + \sin a') \\ &= 0{\cdot}406 \sin a' + 0{\cdot}008 \sin 3a' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.2f)\end{aligned}$$

5.3. Lunar motions

In the case of the moon we have additional terms because the moon oscillates about the ecliptic, so that we have the latitude (Fig. 5.1) to take into consideration.

Let

r = the distance of the moon ;
c = its mean distance ;
x = the longitude of the moon ;
y = the latitude of the moon ;
s = the mean longitude of the moon, increasing by $0°{\cdot}5490$ per mean solar hour ;
p = the mean longitude of the lunar perigee, increasing by $0°{\cdot}0046$ per mean solar hour ;
N = the mean longitude of the ascending node, increasing by $-0°{\cdot}0022$ per mean solar hour ;
a = the right ascension of the moon ;
d = the declination of the moon.

Then the movement of the moon north of the ecliptic is given by

$$\sin y = 0{\cdot}089 \sin (x - \mathrm{N}) \quad . \quad . \quad . \quad . \quad (5.3a)$$

and again we can test this formula. Thus at $x = \mathrm{N}$ the moon, by definition of N, is at the ascending node and its latitude is zero. The maximum latitude is known to be 5° 8′ (apart from very small variations which we shall ignore), and the sine of this angle is 0·089.

Then we have

$$c/r = 1 + 0{\cdot}055 \cos(s - p) + 0{\cdot}010 \cos(s - 2h + p) + 0{\cdot}008 \cos(2s - 2h) + 0{\cdot}003 \cos(2s - 2p) \quad (5.3b)$$

$$x = s + 0{\cdot}110 \sin(s - p) + 0{\cdot}022 \sin(s - 2h + p) + 0{\cdot}011 \sin(2s - 2h) + 0{\cdot}004 \sin(2s - 2p) \quad (5.3c)$$

and the first two terms in each can be discussed as in the case of the sun. The value of the parallax is greatest, so far as these two terms are concerned, when the moon is in mean perigee, and we should expect the actual moon to be near actual perigee when the mean moon is at mean perigee. The second term in the longitude shows that the longitude is increasing most rapidly when the moon is at perigee, exactly as we found in the case of the sun. It may be noted that the remaining terms involve the longitude of the sun and actually these terms express perturbations of the lunar motion due to direct attraction by the sun. Thus the parallax is increased when the sun and moon are in line ($s = h$), and when the moon is also in perigee ($s = h = p$).

The lunar declination is related to the right ascension by the formula

$$\sin d = 0{\cdot}406 \sin a + 0{\cdot}008 \sin 3a + 0{\cdot}090 \sin(a - \mathrm{N}) + 0{\cdot}006 \sin(3a - \mathrm{N}) \quad . \quad . \quad (5.3d)$$

What has been said about the solar declination applies to the first two terms of this expression. The last two terms obviously give the variation in declination with the changing latitude of the moon. The principal term of these two is practically equal to the latitude, the formula for which was given earlier. The last term arises in passing from the longitude to the right ascension. This formula can be readily tested for $a = 90°$ and $\mathrm{N} = 0°$, $90°$, $180°$, the corresponding values of d being $28° 35'$, $23° 27'$, and $18° 19'$, as shown in Art. 5.1.

The right ascension is related to the longitude by the formula

$$a = x - 0{\cdot}043 \sin 2s + 0{\cdot}019 \sin \mathrm{N} - 0{\cdot}019 \sin(2s - \mathrm{N}) \quad . \quad . \quad (5.3e)$$

The first two terms are similar to those for the solar right ascension; the last two terms give corrections for the latitude of the moon, and we see that the terms cancel one another, as they should, when $s = \mathrm{N}$, for then the mean moon is at the node, and is therefore on the ecliptic, so that the latitude is zero.

The remarks made above should be sufficient to justify the formulæ, but the student can test the degree of accuracy, if he so desires, by a few computed comparisons with the values given in the Nautical Almanac.

These formulæ have been given to facilitate some reasoning given in the next chapter by which the derivation of the harmonic constituents can be explained, using general, non-mathematical, methods.

5.4. Relation between angles Z and a, Z′ and a′

We now have to consider the relation between the angle Z and the right ascension a. In Fig. 5.3, let the circle denote the celestial equator, and ♈, X, M the intersections with it of the meridian planes through ♈, the place X and the moon. A complete

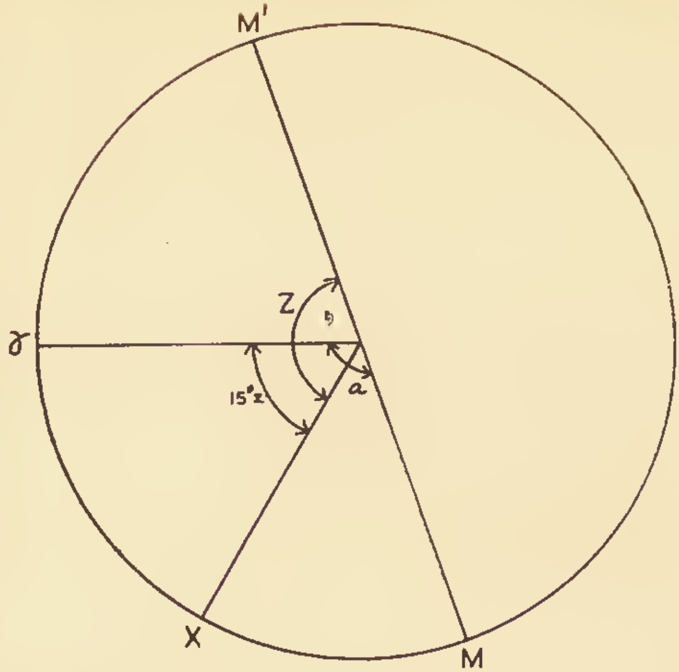

FIG. 5.3. Relations between angles 15°z, a, and Z.

revolution of the earth relative to the fixed point ♈ takes place in one sidereal day, which is 23 hours 56 minutes 4 seconds, expressed in solar time. We shall take

$$z = \text{local sidereal time}$$
$$= \text{the angle } ♈X \text{ (in degrees), divided by 15.}$$

Then, since Z is the angle between the meridians through X and M′, we have

$$a - 15^\circ z = 180^\circ - Z$$

or

$$Z = 15^\circ z - a + 180^\circ \qquad (5.4a)$$

Similarly in the corresponding case of the solar angles we have

$$Z' = 15^\circ z - a' + 180^\circ \qquad (5.4b)$$

CHAPTER VI

DESCRIPTION OF HARMONIC CONSTITUENTS

6.1. Harmonic terms, simple and compound

It will now be evident that the expressions for the equilibrium tide, though given in the forms of trigonometrical expressions involving Z and d, are more complex than appear at first sight. It is our object to resolve these expressions into "simple harmonics," in which the angles of the trigonometrical functions change uniformly with time and in which the amplitudes are invariable, or can be considered as invariable for a lengthy period of time.

In general, a simple harmonic term can be expressed in the form

$$R \cos (nt - k)$$

where R = the amplitude;
t = the time;
n = the speed, or the increment in angle per unit of time;
k = the "phase-lag";
$nt - k$ = the argument;
$360°/n$ = the period of the oscillation (in hours, if speeds are measured in degrees per hour).

In order to avoid continual repetition of "degrees per mean solar hour" we find it convenient to define

the *speed number* = the numerical value of the speed in degrees per mean solar hour.

A compound harmonic is one in which any or all of R, n, k are variable, and it is generally, but not always, possible to express such a compound harmonic term as made up of a number of simple harmonic terms. This is possible in the case of expressions for the tidal forces and elevations. In order to avoid lengthy trigonometrical expansions of these compound harmonics, we shall demonstrate two principles and draw some simple deductions, mainly in a qualitative, rather than a quantitative, way.

(1) Suppose that the amplitude R is a variable quantity and that the variable part itself is a simple harmonic term so that we have, for instance, such an expression as

$$(R_0 + R_1 \cos mt) \cos nt$$

where R_0, R_1, m, n are constants. Then the simple and well-known formula for the product of two cosines gives us the equivalent of the above expression the three simple harmonic terms

$$R_0 \cos nt + \tfrac{1}{2}R_1 \cos (n + m)t + \tfrac{1}{2}R_1 \cos (n - m)t$$

Hence, if the amplitude of a compound harmonic with speed n includes a variable term which can be represented by a simple harmonic term with speed m, the expression can be represented by harmonic terms with speeds

$$n, \quad n + m, \quad \text{and } n - m.$$

This is an important result, and similar expressions involving sines yield the same fundamental result.

(2) In the example just given only the *amplitude* of the term is varied. Consider, however, the first two of the three harmonic terms given above:—

$$R_0 \cos nt + \tfrac{1}{2}R_1 \cos (n + m)t$$

High water occurs at $t = 0$, and then the compound is not behaving like $\cos nt$ nor like $\cos (n + m)t$, but like an oscillation with a speed between the two, if R_1 is small compared with R_0. In other words, the speed is somewhat greater than n at high water (if m is positive) and the amplitude is greater than R.

If, however, m is negative, then while the amplitude is $R_0 + \frac{1}{2}R_1$ near $t = 0$, the function is behaving like a harmonic oscillation with speed less than n.

6.2. Harmonic constituents of the lunar equilibrium semidiurnal tide

The lunar equilibrium semidiurnal tide varies as

$$(c/r)^3 \cos^2 d \cos 2Z \quad . \quad . \quad . \quad . \qquad \text{(see 4.3d)}$$

with c as the mean value of r.

Suppose that the moon moves at its mean speed and at its mean distance, in the plane of the equator, so that $d = 0$, $c/r = 1$. Then the semidiurnal tide is simply proportional to $\cos 2Z$, and as the angle $2Z$ increases under these assumptions at a constant rate we have a simple harmonic oscillation, with a period of half a lunar day; that is, 12·42 mean solar hours. The speed-number is thus 360/12·42. This mean tide is denoted by M_2, so that we have

M_2 = the principal lunar semidiurnal constituent, with speed = 28°·984 per mean solar hour.

Now suppose that the moon's distance is not constant but varies as in actuality with a period of 27·555 days or 661·3 hours. The speed-number of the harmonic term expressing this variation is thus 0·544 (= 360/661·3). The changing distance of the moon (r) affects the amplitude of the lunar tide so that we have to modify the simple constituent by means of a factor as in case (1) of the previous article; hence we get, in addition to M_2, two other simple harmonic constituents with speed-numbers equal to

$$28{\cdot}984 + 0{\cdot}544 = 29{\cdot}528$$

and

$$28{\cdot}984 - 0{\cdot}544 = 28{\cdot}440$$

As we saw from case (1) of Art. 6.1 these constituents would have equal amplitudes (denoted in the example of case (1) by $\frac{1}{2}R_1$) if only the amplitude of the main term M_2 were modified.

But the changing distance of the moon affects its speed in the orbit. As stated in Art. 5.2, at great distances the attractive forces are less, and the rate of movement is less, while at its nearest distance the moon is moving most rapidly. Now from Fig. 5.3 we see that while the place X is moving eastwards because of the rotation of the earth, the angle Z depends upon the distance between the meridian of lunar transit and the meridian of X. Therefore a movement of the moon which is more rapid than usual means that Z is not increasing so rapidly as usual; that is, the speed of the complex harmonic term is diminished when the tide is greatest. Case (2) of the previous article would thus indicate that the principal term acting with M_2 is the one with the lesser of the two speeds given above, so that the two constituents given above as due to motion in an elliptic orbit are defined as

N_2 = the larger lunar elliptic semidiurnal constituent, with speed = 28°·440 per mean solar hour.
L_2 = the smaller lunar elliptic semidiurnal constituent, with speed = 29°·528 per mean solar hour.

We have assumed in the above explanation that the variations in the distance of the moon can be expressed by a simple harmonic term with speed-number 0·544, but formulæ (5·3b) and (5.3c) show that the actual motion involves other harmonic variations, each of which will generate two harmonic tidal constituents. Thus while the speed-number of the first term with argument $(s - p)$ is 0·544, the term with argument $(s - 2h + p)$ has a speed-number of 0·471 (since, according to Arts. 5.2 and 5.3 the rates of increase of s, h, p are respectively 0°·5490, 0°·0411 and 0°·0046

per mean solar hour), and we deduce, exactly in the same way as for the constituents N_2 and L_2, the two constituents

ν_2 with speed $= 28°{\cdot}513$ per mean solar hour ;
λ_2 with speed $= 29°{\cdot}455$ per mean solar hour.

The next terms in formulæ (5.3b) and (5.3c) have arguments of $(2s - 2h)$ and $(2s - 2p)$, with speed-numbers 1·016 and 1·089, respectively, from which we deduce as above the constituents

μ_2 with speed $= 27°{\cdot}968$ per mean solar hour ;
S_2 with speed $= 30°{\cdot}000$ per mean solar hour ;
$2N_2$ with speed $= 27°{\cdot}895$ per mean solar hour ;
and one with speed $= 30°{\cdot}073$ per mean solar hour.

From general reasoning on the same lines as that already given, the relative importance of these terms could be estimated. The last-mentioned constituent is negligible, and that denoted by S_2, as here derived, is, of course, not the principal solar semidiurnal constituent, but a small lunar perturbation of the solar constituent.

We now have to consider the effects of declination. In the semidiurnal tide the declinational factor is $\cos^2 d$. Now suppose that the moon moves only in the ecliptic ; then the declination completes a cycle of oscillation, north and south of the equator, in exactly the same period as the revolution of the moon in its orbit, that is, in 27·3216 mean solar days, or in 655·7 mean solar hours, and, since $2\cos^2 d = 1 + \cos 2d$, the periodic term $\cos 2d$ has a cycle of oscillation in 327·85 hours and thus a speed-number of 1·098 (= 360/327·85). Hence this variable factor in the lunar amplitude will give two harmonic terms in association with the principal term M_2, and their speed-numbers will be the speed-number of M_2 plus or minus 1·098 ; that is, the terms in question will have speed-numbers equal to

$$28{\cdot}984 + 1{\cdot}098 = 30{\cdot}082$$

and

$$28{\cdot}984 - 1{\cdot}098 = 27{\cdot}886$$

Now if the declination did not affect the movement in right ascension, these two terms would be equal, and we have thus to inquire how the speed of the complex harmonic constituent representing the tide is affected by declination. It was shown in Art. 5.3 that when the moon is at ♈ (i.e. has zero declination) then the rate of change in right ascension is less than the average. Consequently, the angle Z in Fig. 5.3 and in formula (5.3f) is changing more rapidly than usual and therefore (see the arguments of Art. 6.1) the speed of the complex harmonic representing the tide is greater than the average speed. Thus we find from Case (2) of Art. 6.1 that the more important of the two constituents with speed-numbers 30·082 and 27·886 must be that with the greater speed. Thus we obtain a constituent

K_2 = the lunar declinational semidiurnal constituent, with speed $= 30°{\cdot}082$ per mean solar hour.

The changes in declination due in the period of 18·61 years will be discussed later, and we shall have further remarks to make about K_2 in the next article.

Many other constituents could be indicated by these methods, but their amplitudes are small, and the constituents derived as above are the ones usually taken into account.

6.3. Harmonic constituents of the solar equilibrium semidiurnal tide

The harmonic constituents of the solar semidiurnal tide are derived from

$$(c'/r')^3 \cos^2 d' \cos 2Z' \quad . \quad . \quad . \quad . \quad \text{(see 4.3d)}$$

where the dashes denote solar values of distance, declination, and the angular distance between the meridian of lower transit of the sun and the meridian of the place X (as in Fig. 5.2 for the corresponding case of the moon).

The cycles of change in parallax and parallax factor are completed in 365·24 days, and thus the harmonic term representing the parallax factor has a speed-number of 0·041 (= 360/8765·8), but while the cycle of change in declination is also annual the cycle of change in the declination factor is semi-annual, since $2\cos^2 d' = 1 + \cos 2d'$, so that the speed-number of the harmonic term for this factor is 0·082, twice that of the parallax term.

If the sun were supposed to move in the equator, at a constant rate, and at a constant distance, then the tide would vary as $\cos 2Z'$, and $2Z'$ would increase at a steady rate of 30° per mean solar hour. Hence we get

S_2 = the principal solar semidiurnal constituent with a speed = 30°·000 per mean solar hour.

The effects of the mean parallax factor are represented by two constituents (analogous to L_2 and N_2) with speed-numbers

30·041 and 29·959.

For reasons exactly similar to those pointing out N_2 as more important than L_2 we deduce that the term with the smaller speed-number is the more important of these two. Hence we get

T_2 = the principal solar elliptic semidiurnal constituent with speed = 29°·959 per mean solar hour.

The declinational factor applied to S_2 also gives terms with speed-numbers

30·082 and 29·918

and exactly as in the case of the lunar tides the one with the greater speed is the more important. This constituent has exactly the same speed as the lunar declinational tide K_2. Hence we define

K_2 = the luni-solar declinational semidiurnal constituent, with speed = 30°·082 per mean solar hour.

6.4. Harmonic constituents of the lunar equilibrium diurnal tide

The lunar equilibrium diurnal tide varies as

$$(c/r)^3 \sin 2d \cos Z \quad . \quad . \quad . \quad . \qquad \text{(see 4.3c)}$$

If we proceed as in the case of the semidiurnal tide, then the tide resulting from the moon moving at a constant speed in the plane of the equator and at a constant distance would be simply harmonic with a speed-number of 14·492, but such a tide would have zero amplitude because of the declinational factor being zero.

If, however, we introduce the declinational factor first, then we have a harmonic factor varying with a mean period of 655·7 hours, and a mean speed-number of 0·549 exactly as in the case of the semidiurnal tide. The effect is to give two constituents of equal amplitude with speed-numbers equal to

$$14{\cdot}492 + 0{\cdot}549 = 15{\cdot}041$$

and

$$14{\cdot}492 - 0{\cdot}549 = 13{\cdot}943$$

These two constituents are called

K_1 = a lunar declinational diurnal constituent, with speed = 15°·041 per mean solar hour;

O_1 = a lunar declinational diurnal constituent, with speed = 13°·943 per mean solar hour.

These two are equal in amplitude, because they neutralise one another to give zero tides when the declination of the moon is zero. (It may be noted, however, that more elaborate investigations yield the result that the amplitude of O_1 is a little greater

than that of the lunar K_1.) There are minor constituents due to the effects of declination on the rate of change of Z but we shall ignore these. (Further reference will be made to K_1 in the next article.)

The changing parallax of the moon also introduces new terms. The variation of the parallactic factor is exactly the same as for the semidiurnal tides, the most important harmonic term having a speed-number of 0·544. Thus we get new constituents from each of K_1 and O_1 with speed-numbers equal to

$$15{\cdot}041 + 0{\cdot}544 \text{ or } 15{\cdot}585$$
$$15{\cdot}041 - 0{\cdot}544 \text{ or } 14{\cdot}497$$

and

$$13{\cdot}943 + 0{\cdot}544 \text{ or } 14{\cdot}487$$
$$13{\cdot}943 - 0{\cdot}544 \text{ or } 13{\cdot}399$$

Because the parallactic factor is associated with a change in Z, it follows, exactly in the case of N_2 and L_2 that the terms with the smaller speeds are more important than the constituents with the greater speeds. These new constituents are called

Q_1 = a lunar diurnal constituent, with speed = 13°·399 per mean solar hour.
M_1 = a lunar diurnal constituent, with speed = 14°·492 per mean solar hour.
J_1 = a lunar diurnal constituent, with speed = 15°·585 per mean solar hour.

It may be noted that in the case of M_1 two constituents generated as above have approximately equal speed-numbers (14·497 and 14·487); these two are not easily separated and they are treated as one whose speed is the arithmetic mean of the two speeds, and whose amplitude and argument are slightly variable.

6.5. Harmonic constituents of the solar equilibrium diurnal tide

The solar diurnal tide varies as

$$(c'/r')^3 \sin 2d' \cos Z' \quad . \quad . \quad . \quad \text{(see 4.3c)}$$

and again it can be shown that sin $2d'$ varies principally with a period which is the same as that of d', namely, 365·24 days. With the sun in the equator then the solar diurnal tide would have a period of 24 mean solar hours and a speed-number of 15·000, but its amplitude would be zero. Bringing in the declinational factor gives two constituents with speed-numbers

$$15{\cdot}000 + 0{\cdot}041 = 15{\cdot}041$$

and

$$15{\cdot}000 - 0{\cdot}041 = 14{\cdot}959;$$

the amplitudes of these, at this stage, are necessarily equal to give zero amplitude when $d' = 0$. It will be noted as in the case of K_2 that the former of these two constituents has a speed-number equal to that of the lunar declinational diurnal constituent K_1, so that we have the two forming

K_1 = the luni-solar declinational diurnal constituent, with speed = 15°·041 per mean solar hour.

The other constituent is known as

P_1 = the solar declinational diurnal constituent, with speed = 14°·959 per mean solar hour.

The terms resulting from the parallax factor on the two constituents K_1 and P_1 are small and usually neglected.

6.6. Harmonic constituents for the equilibrium long period tide

The lunar equilibrium long period tide varies as

$$(c/r)^3 (\tfrac{1}{3} - \sin^2 d) = (c/r)^3 (\tfrac{1}{2} \cos 2d - \tfrac{1}{6}) \quad . \quad . \quad \text{(see 4.3b)}$$

and since the lunar declination varies in a mean period of 655·7 hours then the variable part of the declinational factor can be represented by a harmonic term with speed-number equal to 1·098 (= 360/327·85), and denoted by

Mf = the lunar fortnightly constituent, with speed = 1°·098 per mean solar hour.

The variation of the parallactic term, as in Art. 6.2, involves a harmonic term with speed-number 0·544, and the constant terms in the declinational factor thus give a constituent denoted by

Mm = the lunar monthly harmonic constituent, with speed = 0°·544 per mean solar hour.

Smaller constituents, of course, arise by the interaction of variable parts of the parallax factor on Mf, but these are negligible.

For the solar constituents, the variable part of the declinational factor, which has a period of half a year, simply gives a semi-annual constituent, while the parallax factor gives an annual constituent, so that we have

Sa = the solar annual constituent, with speed = 0°·041 per mean solar hour.
Ssa = the solar semi-annual constituent, with speed = 0°·082 per mean solar hour.

6.7. Nineteen-yearly variations

The orbit of the moon slowly changes in a period of 18·61 years, as was shown in Art. 5.1, and consequently the declination, and all factors depending upon it, vary with the same period. Instead of introducing new constituents it is customary to apply a factor (f), and an increment in the phase (u) to make allowance for these changes. The values of f and u are not the same for all lunar constituents, but as their values can only be accurately determined by mathematical methods, it is sufficient to point out that all the constituents can be written in the form

$$fR \cos (V + u) \qquad \text{(6.7a)}$$

where f is a factor varying in a period of 18·61 years ;
u is an angle varying in a period of 18·61 years ;
V is an angle changing steadily at the mean speed of the constituent ;
R is the amplitude of the constituent.

For the solar constituents the value of f is unity and u is zero. The luni-solar constituents, of course, have values of f and u derived from those pertaining to the lunar parts.

6.8. Definition of harmonic tidal constants

It was stated on page 30 that the equilibrium tide is used as a standard of reference for the tide as it exists, and on page 4 we have referred to Laplace's principle, which indicates that a tidal constituent may be related to the corresponding equilibrium constituent by expressing the tidal constituent in the form

$$fH \cos (V + u - \kappa) \qquad \text{(6.8a)}$$

where f, V, u are defined as in (6.7a)
H is the amplitude of the tidal constituent
κ is the lag of the phase of the tidal constituent behind the phase of the corresponding equilibrium constituent.

H and κ are said to be the *harmonic constants* of the tidal constituent.

CHAPTER VII

HARMONIC TIDAL CONSTITUENTS

THIS chapter will be largely concerned with the details of the mathematical processes necessary to obtain more precise data concerning the harmonic constituents. The general reader will probably be content with the exposition given in the previous chapter and therefore to accept the results of the mathematical processes. These results are given first, together with certain formulæ necessary for the determination of the elements of the lunar and solar orbits, which are required for the arguments of the harmonic constituents. Formulæ are also given for the nodal factors and phase-shifts, together with details of the adjustment of phases for working in standard time. Afterwards the mathematical development is expounded.

(In addition to the harmonic constituents indicated by the development of the equilibrium tide, there are many constituents indicated by the theory of tidal motion in shallow water, for which reference should be made to Chapter VIII.)

7.1. List of harmonic constituents

In the following table the constituents are given as cosines of certain angles called the *arguments*, which involve the time and the lunar and solar *orbital elements*, as well as constants. The time t is taken as Greenwich Mean Time in the table, but rules are given for the use of any standard time in Art. 7.3. The orbital elements are the mean longitudes s, h, p, p', and N, as defined in Arts. 5.2 and 5.3. The constituents are denoted by symbols, but they are not otherwise described, as the descriptions have been sufficiently given in the previous chapter. The speed-numbers give the speeds, or the increments in angle in degrees per mean solar hour, and they are only given to 4 places of decimals, but more accurate values can be obtained from the formulæ for the orbital elements given in Table 7.2.

The abbreviated list of harmonic constituents given below is extracted from a complete list obtained by elaborate mathematical methods, and the relative coefficients are thus a little more exact than is to be expected from the investigations given later in the chapter. The lists and formulæ quoted in Tables 7.1 to 7.3 are extracted from papers by Doodson (see References at the end of the Manual).

TABLE 7.1

List of Harmonic Constituents of Equilibrium Tide on Meridian of Greenwich

Symbol	Argument	Speed number	Relative coefficient
Sa	h	0·0411	0·012
Ssa	$2h$	0·0821	0·073
Mm	$s - p$	0·5444	0·083
MSf	$2s - 2h$	1·0159	0·014
Mf	$2s$	1·0980	0·156
K_1	$15°t + h + 90°$	15·0411	0·531
O_1	$15°t + h - 2s - 90°$	13·9430	0·377
P_1	$15°t - h - 90°$	14·9589	0·176
Q_1	$15°t + h - 3s + p - 90°$	13·3987	0·072
M_1	$15°t + h - s + 90°$	14·4921	0·040
J_1	$15°t + h + s - p + 90°$	15·5854	0·030

TABLE 7.1—*continued.*

Symbol	Argument	Speed number	Relative coefficient
M_2	$30°t + 2h - 2s$	28·9841	0·908
S_2	$30°t$	30·0000	0·423
N_2	$30°t + 2h - 3s + p$	28·4397	0·174
K_2	$30°t + 2h$	30·0821	0·115
ν_2	$30°t + 4h - 3s - p$	28·5126	0·033
μ_2	$30°t + 4h - 4s$	27·9682	0·028
L_2	$30°t + 2h - s - p + 180°$	29·5285	0·026
T_2	$30°t - h + p'$	29·9589	0·025
$2N_2$	$30°t + 2h - 4s + 2p$	27·8954	0·023

General coefficient :— $\frac{3}{2}\frac{M}{E}\frac{e^3}{c^3}e$

Latitude coefficients :—

long period tide $\frac{3}{4}(\frac{1}{3} - \sin^2 l)$

diurnal tide $\frac{1}{2}\sin 2l$

semidiurnal tide $\frac{1}{2}\cos^2 l$

Arguments for constituents at place L° west of Greenwich :—

add $-$ L° to arguments of diurnal constituents.

add $-$ 2L° to arguments of semidiurnal constituents.

(See Art. 7.3 and Art. 7.11.)

The constituents S*a* and S*sa*, though they are indicated by the equilibrium tide, are principally due in nature to meteorological changes in sea level, and the constituent MS*f* in nature is principally due to shallow-water effects.

The constituents K_1 and K_2 are derived partly from the lunar tide and partly from the solar tide.

The developments of the long-period tides indicate two matters of some interest which may be referred to here. The lunar development indicates a term with argument (N + 180°) which represents a nineteen-yearly variation in sea level (see Table 7.12). Its exact relative coefficient is 0·066. Both the lunar and solar tides yield constant multiples of the product of the general coefficient and the latitude coefficient for the long-period tides (relative coefficients 0·505 and 0·234), and these represent contributions from the tidal forces to the figure of the earth.

TABLE 7.2

Orbital Elements at Zero Hour, G.M.T.

Y = the year

D = the number of days elapsed since January 1st in the year Y

i = the integral part of 0·25 (Y − 1901)

= the number of leap years between 1900 and the year Y, excluding Y, as the leap day in this year is counted in D.

$$s = 277°{\cdot}02 + 129°{\cdot}3848\,(Y - 1900) + 13°{\cdot}1764\,(D + i)$$
$$h = 280°{\cdot}19 - 0°{\cdot}2387\,(Y - 1900) + 0°{\cdot}9857\,(D + i)$$
$$p = 334°{\cdot}39 + 40°{\cdot}6625\,(Y - 1900) + 0°{\cdot}1114\,(D + i)$$
$$N = 259°{\cdot}16 - 19°{\cdot}3282\,(Y - 1900) - 0°{\cdot}0530\,(D + i)$$
$$p' = 282°{\cdot}00 \text{ for the century 1900 to 2000.}$$

The increments in the angles per mean solar day are the coefficients of (D + i) and the increments in a year of 365 mean solar days are the coefficients of (Y − 1900).

7.2. Formulæ for nodal variations

The values of f and u (see Art. 6.7) are given by the following formulæ :—

TABLE 7.3

Values of f and u

$$\begin{aligned}
&Mm : f = 1{\cdot}000 - 0{\cdot}130 \cos N\\
&Mf : f = 1{\cdot}043 + 0{\cdot}414 \cos N\\
&K_1 : f = 1{\cdot}006 + 0{\cdot}115 \cos N - 0{\cdot}009 \cos 2N\\
&O_1 : f = 1{\cdot}009 + 0{\cdot}187 \cos N - 0{\cdot}015 \cos 2N\\
&J_1 : f = 1{\cdot}013 + 0{\cdot}168 \cos N - 0{\cdot}017 \cos 2N\\
&M_2 : f = 1{\cdot}000 - 0{\cdot}037 \cos N\\
&K_2 : f = 1{\cdot}024 + 0{\cdot}286 \cos N + 0{\cdot}008 \cos 2N
\end{aligned}$$

$$\begin{aligned}
&Mm : u = 0^\circ{\cdot}0\\
&Mf : u = -\ 23^\circ{\cdot}7 \sin N + 2^\circ{\cdot}7 \sin 2N - 0^\circ{\cdot}4 \sin 3N\\
&K_1 : u = -\ 8^\circ{\cdot}9 \sin N + 0^\circ{\cdot}7 \sin 2N\\
&O_1 : u = 10^\circ{\cdot}8 \sin N - 1^\circ{\cdot}3 \sin 2N + 0^\circ{\cdot}2 \sin 3N\\
&J_1 : u = -\ 12^\circ{\cdot}9 \sin N + 1^\circ{\cdot}3 \sin 2N - 0^\circ{\cdot}2 \sin 3N\\
&M_2 : u = -\ 2^\circ{\cdot}1 \sin N\\
&K_2 : u = -\ 17^\circ{\cdot}7 \sin N + 0^\circ{\cdot}7 \sin 2N
\end{aligned}$$

Q_1 : the values of f and u are the same as for O_1.

N_2, ν_2, μ_2, $2N_2$: the values of f and u are the same as for M_2.

L_2 : the values of f and u are obtained from

$$f \cos u = 1{\cdot}00 - 0{\cdot}25 \cos 2p - 0{\cdot}11 \cos (2p - N) - 0{\cdot}02 \cos (2p - 2N) - 0{\cdot}04 \cos N.$$
$$f \sin u = -0{\cdot}25 \sin 2p - 0{\cdot}11 \sin (2p - N) - 0{\cdot}02 \sin (2p - 2N) - 0{\cdot}04 \sin N.$$

M_1 : the values of f and u are obtained from

$$f \cos u = 2 \cos p + 0{\cdot}4 \cos (p - N).$$
$$f \sin u = \sin p + 0{\cdot}2 \sin (p - N).$$

7.3. Definition of phase-lags

Having defined as in Art. 7.1 the argument of a harmonic constituent of the equilibrium tide as for Greenwich, it is a simple matter to obtain it for any other place. From Art. 4.3 it is evident that the principal part of the variation of the equilibrium tide depends upon the angle Z as defined in Art. 3.4, and if the longitude of the specified place west of Greenwich is L, then clearly the angle Z pertaining to Greenwich will have to be replaced by Z − L. For the semidiurnal constituents, we replace 2Z by 2Z − 2L.

Let

E = the phase of an equilibrium constituent at Greenwich . (7.3a)

and

j = the species number, 0 for long period constituents, 1 for diurnal constituents, 2 for semidiurnal constituents, etc. . . (7.3b)

Therefore

E − jL = the phase of the equilibrium constituent on a meridian L° west of Greenwich . . . . (7.3c)

If the time t is to be expressed in standard time S hours later than Greenwich, we must take

E − jL + nS = the phase of the equilibrium constituent on a meridian L° west of Greenwich, with the time in standard time, S hours later, than Greenwich . . . . (7.3d)

where

n = the speed of the constituent in degrees per mean solar hour (7.3e)

The phase of the corresponding tidal constituent will be defined as $(E - jL + nS) - \kappa$, where

$\kappa =$ the lag of the phase of the tidal constituent behind the phase of the corresponding equilibrium constituent at the place . . . (7.3f)

An alternative procedure, and one which is more convenient in practice, is to express the phase as $E - g$, where

$g =$ the lag of the phase of the tidal constituent behind the phase of the corresponding equilibrium constituent at Greenwich . . . (7.3g)

These two expressions give

$$g = \kappa + jL - nS \qquad (7.3h)$$

It will be apparent that the observations can be related to any specified time meridian, and analysed as though they had been taken at Greenwich. The result will be to give g. The converse process for predictions also uses the Greenwich equilibrium arguments as given in Table 7.1, with the phase-lags g.

*7.4. First stages of development of lunar equilibrium tide

The declinational factors in the lunar equilibrium tide are

$$\cos^2 d, \quad \sin 2d, \quad \tfrac{2}{3} - 2\sin^2 d \qquad \text{(see 4.3d, 4.3c, 4.3b)}$$

or t he semidiurnal, diurnal, and long-period species respectively.

A formula for $\sin d$ is given in (5.3d) in terms of the right ascension and longitude of the node, and it is quite a simple matter to obtain expansions for $\sin^2 d$ by squaring the expression. The operations are considerahly simplified by expressing all sines as cosines, involving only the subtraction of 90° from the arguments, so that we only have to use one trigonometrical formula in all the ensuing work ; viz.

$$\cos A \cos B = \tfrac{1}{2}\cos(A + B) + \tfrac{1}{2}\cos(A - B).$$

The work for $\sin^2 d$ will be written out in full.

From (5.3d) we have

$$\sin d \doteq 0{\cdot}406 \cos(a - 90^\circ) + 0{\cdot}008 \cos(3a - 90^\circ) + 0{\cdot}090 \cos(a - N - 90^\circ) + 0{\cdot}006 \cos(3a - N - 90^\circ)$$

Hence

$$\sin^2 d = 0{\cdot}165 \cos^2(a - 90^\circ) + 0{\cdot}008 \cos^2(a - N - 90^\circ) + \ldots + 2\,[0{\cdot}036 \cos(a-90^\circ)\cos(a-N-90^\circ) + 0{\cdot}003 \cos(a-90^\circ)\cos(3a-90^\circ) + \ldots]$$

in which neglected terms are indicated by $+ \ldots$

This is equivalent to

$$\sin^2 d = 0{\cdot}082\,[1 + \cos(2a - 180^\circ)] + 0{\cdot}004\,[1 + \cos(2a - 2N - 180^\circ)] + 0{\cdot}036\,[\cos(-N) + \cos(2a - N - 180^\circ)] + 0{\cdot}003\,[\cos 2a + \cos(4a - 180^\circ)]$$

in which the terms in square brackets result from the use of the trigonometrical formula given above.

Also,

$$\cos^2 d = 1 - \sin^2 d$$

and since $\sin^2 d$ is a small quantity we can write, by the binomial theorem, with sufficient accuracy,

$$\cos d = (1 - \sin^2 d)^{\frac{1}{2}} = 1 - \tfrac{1}{2}\sin^2 d + \ldots$$

The value of $\sin 2d$ is then simply obtained by multiplying the series for $\sin d$ by $2\cos d$. These and similar results are given below in tabular form.

* See par. 1, page vii.

TABLE 7.4

Expansions of declinational factors in terms of right ascension

(Arguments and Coefficients of Cosines)

$\sin d$		$\sin^2 d$		$\cos d$	
$a - 90°$	0·406	$0°$	0·086	$0°$	0·957
$3a - 90°$	0·008	$2a - 180°$	0·079	$2a$	0·040
$a - N - 90°$	0·090	N	0·036	$N - 180°$	0·018
$3a - N - 90°$	0·006	$2a - N - 180°$	0·036	$2a - N$	0·018
$\cos^2 d$		$\sin 2d$		$\frac{2}{3} - 2\sin^2 d$	
$0°$	0·914	$a - 90°$	0·757	$0°$	0·495
$2a$	0·079	$a - N - 90°$	0·158	$2a$	0·158
$N - 180°$	0·036	$3a - 90°$	0·031	$N - 180°$	0·072
$2a - N$	0·036	$3a - N - 90°$	0·022	$2a - N$	0·072
		$a + N + 90°$	0·011		

We now proceed to combine the declinational factors with the factors involving the time through the angle Z. The diurnal and semidiurnal tides, as indicated by (4.3c) and (4.3d), are proportional to

$$-\sin 2d \cos Z \quad \text{and} \quad \cos^2 d \cos 2Z$$

and the angle Z involves the sidereal time z and the right ascension a according to the formula

$$Z = 15°z - a + 180° \qquad \text{(see 5.4a)}$$

The terms in the expansions of the above products are very simply obtained by means of the standard trigonometrical formula, and are written down at once from Table 7.4 by merely adding the arguments of that table to the angle Z and also by subtracting them from Z. Thus every term in sin $2d$, for instance, yields two terms in the required product. The coefficients of Table 7.4 are simply halved, and the results are given in Table 7.5. Each of the terms in the lower half of the table represents a nodal variation on a term in the upper half of the table.

TABLE 7.5

Expansions in Terms of Sidereal Time and Right Ascension

(Arguments and Coefficients of Cosines)

$-\sin 2d \cos Z$		$\cos^2 d \cos 2Z$	
$15°z - 90°$	0·379	$30°z - 2a$	0·914
$15°z - 2a + 90°$	0·379	$30°z$	0·040
$15°z + 2a - 90°$	0·015	$30°z - 4a$	0·040
$15°z - 4a + 90°$	0·015		
$15°z - N - 90°$	0·079	$30°z - N$	0·018
$15°z + N - 2a + 90°$	0·079	$30°z + N - 4a$	0·018
$15°z - N + 2a - 90°$	0·011	$30°z + N - 2a - 180°$	0·018
$15°z + N - 4a + 90°$	0·011	$30°z - N - 2a + 180°$	0·018
$15°z + N + 90°$	0·006		
$15°z - N - 2a - 90°$	0·006		

*7.5. Second stage of development of lunar equilibrium tide

Since the right ascension a does not change uniformly with the time, the next process is to express it in terms of mean longitudes, and then to replace it by these expressions in the terms of Table 7.5.

* See par. 1, page vii.

It will be noted from Table 7.5 that the principal terms involving a are those with arguments

$$15°z - 2a + 90°,\ 15°z + N - 2a + 90°,\ 30°z - 2a$$

and that these are all of the general form $(A - 2a)$ where A is

$$15°z + 90°,\ 15°z + N + 90°,\ 30°z \text{ respectively.}$$

In all other cases we can replace a by the mean longitude s, but in these three special cases we have to determine an appropriate expression for $\cos(A - 2a)$. We can write

$$\begin{aligned}\cos(A - 2a) &= \cos\{(A - 2s) - (2a - 2s)\} \\ &= \cos(A - 2s)\cos(2a - 2s) + \sin(A - 2s)\sin(2a - 2s) \\ &= \cos(A - 2s)\cos(2a - 2s) + \cos(A - 2s - 90°)\sin(2a - 2s)\end{aligned} \qquad (7.5a)$$

Now the angle $(2a - 2s)$ is given by (5.3c) and (5.3e), and it is small enough for us to be able to write

$$\sin(2a - 2s) = 2a - 2s, \quad \text{and} \quad \cos(2a - 2s) = 1 - \tfrac{1}{2}\sin^2(2a - 2s)$$

exactly as in the computations for Table 7.4. We thus derive expansions given in Table 7.6.

TABLE 7.6

Functions of Right Ascension in Terms of Orbital Elements

(Arguments and Coefficients of Cosines)

$\sin(2a - 2s)$		$\sin^2(2a - 2s)$		$\cos(2a - 2s)$	
$s - p - 90°$	0·220	$0°$	0·024	$0°$	0·988
$2s + 90°$	0·086	$2s - 2p - 180°$	0·024	$2s - 2p$	0·012
$s - 2h + p - 90°$	0·044	$s + p + 180°$	0·019	$s + p$	0·010
$2s - 2h - 90°$	0·022	$3s - p$	0·019	$3s - p - 180°$	0·010
$2s - 2p - 90°$	0·008				
$N - 90°$	0·038				
$2s - N + 90°$	0·038				

From Table 7.6 we can compute Table 7.7, exactly as was done for Table 7.5.

TABLE 7.7

Expansions for Component Parts of cos (A − 2a)

(Arguments and Coefficients of Cosines)

$\cos(A - 2s)\cos(2a - 2s)$		$\cos(A - 2s - 90°)\sin(2a - 2s)$	
$A - 2s$	0·988	$A - 3s + p$	0·110
$A - 4s + 2p$	0·006	$A - s - p - 180°$	0·110
$A - 2p$	0·006	$A - 4s - 180°$	0·043
$A - s + p$	0·005	A	0·043
$A - 3s - p$	0·005	$A - 3s + 2h - p$	0·022
$A + s - p - 180°$	0·005	$A - s - 2h + p - 180°$	0·022
$A - 5s + p + 180°$	0·005	$A - 4s + 2h$	0·011
		$A - 2h - 180°$	0·011
		$A - 4s + 2p$	0·004
		$A - 2p - 180°$	0·004
		$A - 2s - N$	0·019
		$A - 2s + N - 180°$	0·019
		$A - 4s + N - 180°$	0·019
		$A - N$	0·019

The entries in Table 7.7 can now be added together, where terms have the same argument, remembering that terms differing in argument by 180° have really opposite signs, and that the result is the expansion for cos $(A - 2a)$ given in Table 7.8. In this table certain small terms have been omitted as they do not contribute to later results.

TABLE 7.8

Expansion of cos (A − 2*a*) in Terms of Orbital Elements

(Arguments and Coefficients of Cosines)

$A - 2s$	0·988	$A - 2s - N$	0·019
$A - 3s + p$	0·110	$A - 2s + N - 180°$	0·019
$A - s - p - 180°$	0·110	$A - 4s + N - 180°$	0·019
$A - 4s - 180°$	0·043	$A - N$	0·019
A	0·043		
$A - 3s + 2h - p$	0·022		
$A - s - 2h + p - 180°$	0·022		
$A - 4s + 2h$	0·011		
$A - 2h - 180°$	0·011		
$A - 4s + 2p$	0·010		

We are now in a position to return to Table 7.5, and to expand the three principal terms which involve a, as noted at the commencement of this article. The appropriate procedure is to write in Table 7.8 the values of A as

$$15°z + 90°,\ \ 15°z + N + 90°,\ \ 30°z \ \ \text{respectively}$$

and to multiply the results by the coefficients

$$0{\cdot}379,\ \ 0{\cdot}079,\ \ 0{\cdot}914 \ \text{respectively.}$$

The resulting terms have then to be combined with the remaining terms of Table 7.5 in which we simply write $a = s$, and the final results are given in Table 7.9.

TABLE 7.9

Expansions in Terms of Sidereal Time and Orbital Elements

(Arguments and Coefficients of Cosines)

$-\sin 2d \cos Z$		$\cos^2 d \cos 2Z$	
$15°z - 2s + 90°$	0·376	$30°z - 2s$	0·903
$15°z - 90°$	0·361	$30°z - 3s + p$	0·101
$15°z - 3s + p + 90°$	0·042	$30°z - s - p - 180°$	0·101
$15°z - s - p - 90°$	0·042	$30°z$	0·079
$15°z + 2s - 90°$	0·015	$30°z - 3s + 2h - p$	0·020
$15°z - 3s + 2h - p + 90°$	0·008	$30°z - s - 2h + p - 180°$	0·020
$15°z - s - 2h + p - 90°$	0·008	$30°z - 4s + 2h$	0·010
		$30°z - 2h - 180°$	0·010
$15°z - N - 90°$	0·072	$30°z - 4s + 2p$	0·009
$15°z + N - 2s + 90°$	0·071		
$15°z - N + 2s - 90°$	0·011	$30°z + N - 2s - 180°$	0·035
$15°z + N - 3s + p + 90°$	0·009	$30°z - N$	0·035
$15°z + N - s - p - 90°$	0·009		
$15°z + N + 90°$	0·009		

*7.6. Third stage of development of lunar equilibrium tide

It is now necessary to apply the parallax factor common to all lunar species of tides. This is derived from (5.3b) :—

$$(c/r)^3 = 1 + 0{\cdot}165 \cos(s - p) + 0{\cdot}031 \cos(s - 2h + p) + 0{\cdot}027 \cos(2s - 2h) + 0{\cdot}013 \cos(2s - 2p)$$

* See par. 1, page vii.

Only the principal terms from the expansions given in Table 7.9 yield results which are of any importance, and the new terms arising from these are grouped together in Table 7.10. These new terms have to be added to the terms in Table 7.9 to give the final values as in Table 7.11.

The terms in the lower half of Table 7.11 represent nodal variations of the principal terms, and these will be dealt with in Art. 7.8. Certain terms are bracketed together as their arguments differ only by multiples of p or N, whose rates of change are too small to allow the terms to be separable by ordinary methods of analysis.

TABLE 7.10

Parallax Terms Supplementary to the Terms of Table 7.9

(Arguments and Coefficients of Cosines)

$15°z - s - p + 90°$	0·031	$30°z - s - p$	0·075
$15°z - 3s + p + 90°$	0·031	$30°z - 3s + p$	0·075
$15°z - s - 2h + p + 90°$	0·006	$30°z - s - 2h + p$	0·014
$15°z - 3s + 2h - p + 90°$	0·006	$30°z - 3s + 2h - p$	0·014
$15°z + s - p - 90°$	0·030	$30°z - 2h$	0·012
$15°z - s + p - 90°$	0·030	$30°z - 4s + 2h$	0·012
$15°z + s - 2h + p - 90°$	0·006	$30°z - 2p$	0·006
$15°z - s + 2h - p - 90°$	0·006	$30°z - 4s + 2p$	0·006
$15°z + s - p - N - 90°$	0·006	$30°z - 2s$	0·008
$15°z - s + p - N - 90°$	0·006	$30°z - 4s + 2p$	0·008
$15°z - s - p + N + 90°$	0·006	$30°z - 2p - 180°$	0·008
$15°z - 3s + p + N + 90°$	0·006	$30°z - 2s - 180°$	0·008
		$30°z + s - p$	0·007
		$30°z - s + p$	0·007

TABLE 7.11

Harmonic Expansion of Lunar Equilibrium Tide in Terms of Sidereal Time and Orbital Elements

(Arguments and Coefficients of Cosines)

Symbol	Diurnal tide		Symbol	Semidiurnal tide	
O_1	$15°z - 2s + 90°$	0·376	M_2	$30°z - 2s$	0·903
K_1	$15°z - 90°$	0·361	N_2	$30°z - 3s + p$	0·176
Q_1	$15°z - 3s + p + 90°$	0·073	K_2	$30°z$	0·079
M_1 {	$15°z - s + p - 90°$	0·030	ν_2	$30°z - 3s + 2h - p$	0·034
M_1 {	$15°z - s - p - 90°$	0·011	L_2 {	$30°z - s - p + 180°$	0·026
J_1	$15°z + s - p - 90°$	0·030	L_2 {	$30°z - s + p$	0·007
	$15°z + 2s - 90°$	0·015	$2N_2$	$30°z - 4s + 2p$	0·023
	$15°z - 3s + 2h - p + 90°$	0·014	μ_2	$30°z - 4s + 2h$	0·021
{	$15°z - N - 90°$	0·072			
{	$15°z + N + 90°$	0·009		$30°z - 2s + N + 180°$	0·035
	$15°z - 2s + N + 90°$	0·071		$30°z - N$	0·035
	$15°z - 3s + p + N + 90°$	0·015			
	$15°z + 2s - N - 90°$	0·011			

(In order to compare these terms with those given in Table 7.1 it is necessary to convert the sidereal time to mean solar time by formulæ obtained in Art. 7.11.)

It will be noted that two of the terms of the diurnal tide with symbol M_1 are bracketed together. The reason for this is that their arguments differ only by $2p$ and the two terms only separate in angle by about 80° per year so that they are not

separable by the normal processes of analysis. For this reason the two are combined and the values of f and u are dependent upon p as well as N, though normally they are dependent only upon N. A similar grouping of terms is necessary with the constituent L_2.

The constituents K_1 and K_2 have to be combined with the solar constituents of the same speeds.

*7.7. Harmonic development of the lunar long-period tide

The lunar long-period tide is proportional to

$$(c/r)^3 \left(\tfrac{2}{3} - 2 \sin^2 d\right) \quad . \quad . \quad . \quad . \qquad \text{(see 4.3b)}$$

The first procedure is to replace cos $2a$ in Table 7.4 by the expression given in Table 7.8 with $A = 0$, and then to multiply by the expansion for $(c/r)^3$ as given in (5.3b). Only the principal terms need to be considered and the two processes readily give the final results of Table 7.12.

TABLE 7.12

Harmonic Expansion of the Lunar Long-period Tide in Terms of Orbital Elements

(Arguments and Coefficients of Cosines)

Symbol		
—	$0°$	0·502
Mf	$2s$	0·156
Mm	$s - p$	0·082
—	$3s - p$	0·030
—	$N + 180°$	0·069
—	$2s - N$	0·069

*7.8 Formulæ for f and u

The terms whose arguments differ only by multiples of N, and by constants, need to be compounded so as to be expressed in the general form proportional to

$$f \cos (V + u)$$

where V is the argument of the principal term. The procedure will be illustrated for the case of the constituent M_2, which is composed of the two terms

$$0{\cdot}903 \cos (30°z - 2s) + 0{\cdot}035 \cos (30°z - 2s + N + 180°)$$

These can be successively written as

$$\begin{aligned}
&0{\cdot}903 \cos (30°z - 2s) - 0{\cdot}035 \cos (30°z - 2s + N) \\
&= 0{\cdot}903 f \cos u \cos (30°z - 2s) - 0{\cdot}903 f \sin u \sin (30°z - 2s) \\
&= 0{\cdot}903 f \cos (30°z - 2s - u)
\end{aligned}$$

provided that we take

$$\begin{aligned}
f \cos u &= 1 - 0{\cdot}039 \cos N \\
f \sin u &= \quad - 0{\cdot}039 \sin N
\end{aligned}$$

We then have

$$\tan u = \frac{- 0{\cdot}039 \sin N}{1 - 0{\cdot}039 \cos N}$$

and

$$f^2 = 1 - 0{\cdot}078 \cos N + 0{\cdot}002$$

and since u and $(f - 1)$ are small quantities we can then write approximately

$$u = \tan u = - 0{\cdot}039 \sin N$$

and

$$f = 1{\cdot}001 - 0{\cdot}039 \cos N.$$

When u is converted from radians to degrees we get $u = - 2°{\cdot}3 \sin N$

* See par. 1, page vii.

The accurate values are given in Art. 7.2, and it may be remarked that as many of the smaller nodal terms have been discarded in the numerical processes given above all tbe data given in Art. 7.2 cannot be reproduced.

*7.9. Harmonic development of the solar equilibrium tide

The formulæ for the solar motions are much simpler than those for tbe lunar motions, and from (5.2e) we have

$$\sin d' = 0{\cdot}406 \cos (a' - 90^\circ) + 0{\cdot}008 \cos (3a' - 90^\circ)$$

whence

$$\sin^2 d' = 0{\cdot}082 + 0{\cdot}079 \cos (2a' - 180^\circ) \qquad (7.9a)$$

$$\cos^2 d' = 0{\cdot}918 + 0{\cdot}079 \cos 2a' \qquad (7.9b)$$

$$\cos d' = 0{\cdot}959 + 0{\cdot}040 \cos 2a' \qquad (7.9c)$$

$$\sin 2d' = 0{\cdot}764 \cos (a' - 90^\circ) + 0{\cdot}032 \cos (3a' - 90^\circ) \qquad (7.9d)$$

Hence, with

$$Z' = 15^\circ z - a' + 180^\circ \qquad \text{(see 5.4b)}$$

we get results as in Table 7.13.

TABLE 7.13

Expansions in Terms of Sidereal Time and Right Ascension

(Arguments and Coefficients of Cosines)

$-\sin 2d' \cos Z'$		$\cos^2 d' \cos 2Z'$	
$15^\circ z - 90^\circ$	0·382	$30^\circ z - 2a'$	0·918
$15^\circ z - 2a' + 90^\circ$	0·382	$30^\circ z$	0·040
$15^\circ z + 2a' - 90^\circ$	0·016	$30^\circ z - 4a'$	0·040
$15^\circ z - 4a' + 90^\circ$	0·016		

We now substitute for $2a'$ using an expansion derived from (5.2d) and (5.2b)

$$\sin (2a' - 2h) = 2a' - 2h = 0{\cdot}068 \sin (h - p') - 0{\cdot}086 \sin 2h$$
$$= 0{\cdot}068 \cos (h - p' - 90^\circ) + 0{\cdot}086 \cos (2h + 90^\circ) \qquad (7.9e)$$

If we take the approximation

$$\cos (2a' - 2h) = 1{\cdot}000 \qquad (7.9f)$$

then we get as before, the typical formula

$$\cos (A - 2a') = \cos (A - 2h) + \cos (A - 2h - 90^\circ) \sin (2a' - 2h) \qquad (7.9g)$$

and thus obtain the expansion given in Table 7.14.

TABLE 7.14

Expansion of cos (A − 2a′) in Terms of Orbital Elements

(Arguments and Coefficients of Cosines)

$A - 2h$	1·000
A	0·043
$A - 4h + 180^\circ$	0·043
$A - h - p' + 180^\circ$	0·034
$A - 3h + p'$	0·034

We now substitute $A = 15^\circ z + 90^\circ$ in one term, and $A = 30^\circ z$ in another term, in tbe expansions of Table 7.13, while we replace a' by h in all other terms, and the results are collected in Table 7.15.

* See par. 1, page vii.

TABLE 7.15

Expansions in Terms of Sidereal Time and Orbital Elements

(Arguments and Coefficients of Cosines)

$-\sin 2d' \cos Z'$		$\cos^2 d' \cos 2Z'$	
$15°z - 2h + 90°$	0·382	$30°z - 2h$	0·918
$15°z - 90°$	0·366	$30°z$	0·079
$15°z + 2h - 90°$	0·016	$30°z - h - p' + 180°$	0·031
$15°z - h - p' - 90°$	0·013	$30°z - 3h + p'$	0·031
$15°z - 3h + p' + 90°$	0·013		

The final stage is to multiply the above results by the parallax factor

$$(c'/r')^3 = 1{\cdot}000 + 0{\cdot}051 \cos (h - p') \quad \text{(see 5.2a)}$$

and the results are given in Table 7.16.

TABLE 7.16

Harmonic Expansion of Solar Equilibrium Tide in Terms of Sidereal Time and Orbital Elements

(Arguments and Coefficients of Cosines)

Symbol	Diurnal tide		Symbol	Semidiurnal tide	
P_1	$15°z - 2h + 90°$	0·382	S_2	$30°z - 2h$	0·918
K_1	$15°z - 90°$	0·366	K_2	$30°z$	0·079
—	$15°z - 3h + p' + 90°$	0·023	T_2	$30°z - 3h + p'$	0·054
—	$15°z + 2h - 90°$	0·016			

(The coefficients of the terms in this table need to be multiplied by the factor 0·460, as obtained in Art. 2.4, to render them comparable with the coefficients of the lunar terms. For a comparison with the terms given in Table 7.1 it is also necessary to convert the sidereal time to mean solar time by formulæ given in Art. 7.11.)

The constituents K_1 and K_2 have to be combined with the corresponding constituents derived from the lunar tide.

*7.10. Harmonic development of the solar long-period tide

The solar long-period tide is proportional to

$$(c'/r')^3 \left(\tfrac{2}{3} - 2 \sin^2 d'\right) \quad \text{(see 4.3a)}$$

and by (5.2a) and (7.9a) this is equal to

$$\{1 + 0{\cdot}051 \cos (h - p')\} \, (0{\cdot}503 + 0{\cdot}158 \cos 2a') \quad (7.10a)$$

and it is sufficient to replace a' by h. The resulting terms are given in Table 7.17.

TABLE 7.17

Harmonic Expansion of the Solar Long-period Tide in Terms of Orbital Elements

(Arguments and Coefficients of Cosines)

Symbol		
—	$0°$	0·503
Ssa	$2h$	0·158
Sa	$h - p'$	0·026

(In order to compare with the lunar terms, the coefficients need to be multiplied by 0·460.)

* See par. 1, page vii.

*7.11. Arguments in terms of Greenwich Mean Time

The results of the developments of the previous articles have all been expressed in terms of local sidereal time. We now proceed to consider the formula for conversion to Greenwich Mean Time.

Let t = time elapsed in mean solar hours from mean solar midnight at Greenwich . . . . (7.11a)

L = longitude of place west of Greenwich . . . . . (7.11b)

Also in Fig. 7.1 let ♈, S, X, G denote the places of the vernal equinox, the mean sun, the meridian through the place X, and the meridian of Greenwich, respectively, in the equator. Also let S′ denote the meridian at which midnight is being experienced.

Then

$$\left.\begin{aligned} S'G &= 15^\circ t \\ \Upsilon S &= h \\ S'X &= 15^\circ t - L \\ SX &= 15^\circ t - L - 180^\circ \\ \Upsilon X &= \Upsilon S + SX = 15^\circ t + h - L - 180^\circ \end{aligned}\right\} \qquad (7.11c)$$

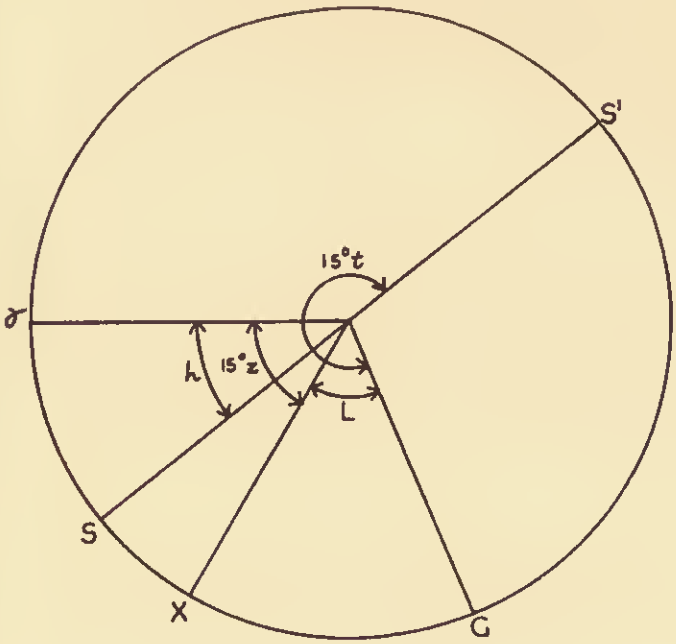

FIG. 7.1. Relations between angles 15° t, 15°z, h and L.

Hence we get

$$15^\circ z = \Upsilon X = 15^\circ t + h - 180^\circ - L \qquad (7.11d)$$

If this is substituted in the expressions obtained in Tables 7.11 and 7.16 we get the arguments given in terms of G.M.T., as in Table 7.1. In the latter table, however, the tabular entries are given for Greenwich meridian so that L is omitted. The correction for any other meridian is given in the footnote to the table. See also Art 7.3.

* See par. 1, page vii.

CHAPTER VIII

SHALLOW-WATER TIDES

8.1. The generation of shallow-water tides

THE influence of terrestrial conditions on any tidal motion generated by forces external to the earth can only be adequately investigated after a consideration of the laws of propagation of "progressive waves" and "standing oscillations," the former in long open channels, and the latter in channels closed at one end. This theory is postponed to Chapters XVII and XVIII, and we shall deal with the theoretical results in so far as they give guidance for the determination of harmonic constituents which express the effects of shallow water upon the astronomically generated tides.

If the depth of the coastal waters were always very great it would not be necessary to consider the distortional effects associated with shallow water, but in actuality the effects of shallow water are important. They are most prominent in estuaries and arms of the sea in which we should normally expect to apply the theory of standing oscillations (see Chapter XVIII) rather than the theory of progressive waves (see Chapter XVII). It is unfortunate, however, that mathematicians have not yet provided a satisfactory solution of the problem of distortion of standing oscillations in shallow water, but sufficient is known to apply empirically the results obtained from a discussion of the distortion of a progressive wave. Further, it happens that in a very shallow estuary the effects of friction are to diminish the reflected waves rather more than the ingoing waves so that in an estuary it often happens that there is an apparent progressive wave.

Suppose, therefore, that in Fig. 8.1 the curve (a) represents the profile of a progressive wave entering a channel from deep water. Such a wave will be represented by a simple harmonic curve in which the time interval from low water to high water is equal to that from high water to low water. It is shown in Art. 17.8 that the effect of travelling along an infinitely long channel in shallow water is to change the shape of the wave so that high water is accelerated and low water is retarded, and that any point in the profile of the wave travels at a rate

$$c = \sqrt{g(h + 3y)}$$

where $c =$ rate of travel of a point in the wave profile
$g =$ the coefficient of gravitational force
$h =$ the mean depth of water in the channel
$y =$ the elevation of the point above the mean level.

For our purposes we can take

$$c = \left(1 + \frac{3}{2}\frac{y}{h}\right)\sqrt{gh} \quad . \quad . \quad . \quad . \quad (8.1a)$$

Suppose that after a certain lapse of time t the profile is again drawn on the same diagram as the original profile, so that the points M at the mean level (Fig. 8.1) are made to coincide. Then the high water H will appear to have moved to H′ and the low water L will have appeared to have moved to L′. In the time t, any point will have actually moved through a distance ct where c depends on y as in (8.1a). Relatively to the point M, however, all other points will appear to have moved through a distance equal to

$$\frac{3}{2}\frac{y}{h}\sqrt{gh} \, . \, t \quad . \quad . \quad . \quad . \quad . \quad (8.1b)$$

Hence the distance RR′ through which R will appear to have moved will be proportional to the elevation at R, and therefore RR′/HH′ will be equal to the ratio of the elevations at R and H. The apparent movement HH′ has been supposed to have become equivalent to a shift of 30° in phase of (a), and the values of RR′ have been computed for a sufficient number of points to draw curve (b).

Now let the elevations for (a) be subtracted from those for (b), and let the result be given in (c). It is at once apparent that the latter curve has two complete oscillations for one of the original curve (a). Now if a wave like (a) passes an observer, the phenomena could be drawn by him on an exactly similar diagram to that of Fig. 8.1, with the time scale drawn to the *left*. Then the curve as given, reading

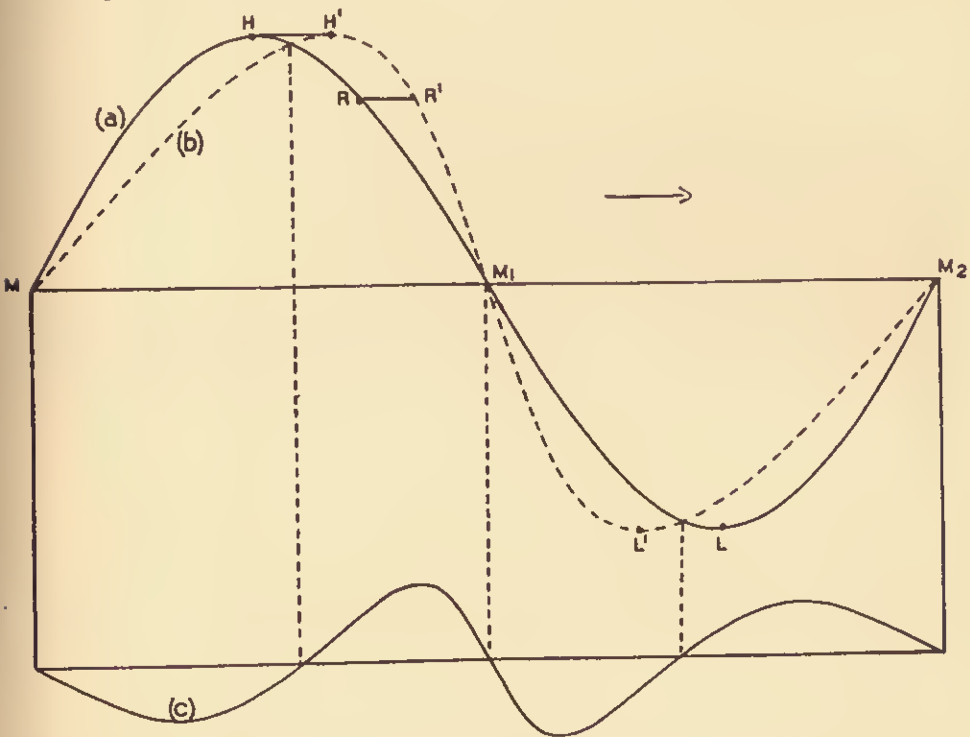

FIG. 8.1. Deduction of quarter-diurnal tide from change of shape of progressive wave.

from right to left, would firstly give low water, and then high water. If (a) represents a tidal oscillation with a period of 12 hours, then (c) will represent an oscillation with a period of six hours.

For the moment, suppose that (c) is a pure harmonic curve, then the meaning of the work we have just done is that the actual tide (b) can be represented by a pure harmonic (a) and a pure harmonic (c), the latter having a period only half the period of the primary curve (a). The additional tide (c) is called a shallow-water tide.

But it may be readily seen that the curve (c) is not a pure harmonic curve, for the distances between the points of zero level are not equal. For the further examination of this curve let it be transferred to Fig. 8.2, where it is still called (c), and let curve (d) be a simple sine curve whose amplitude is the average high water height of curve (c). In exactly the same way as before let the elevations of (d) be subtracted from those of (c), and let the results be graphed as in (e). This resulting curve has three oscillations for one in (a), so that if (a) represents a semidiurnal oscillation then (e) represents a sixth-diurnal oscillation. This process could be continued to reveal eighth-diurnal tides and even higher species of tides.

Hence we conclude that any tide upon the earth may be expected to contain

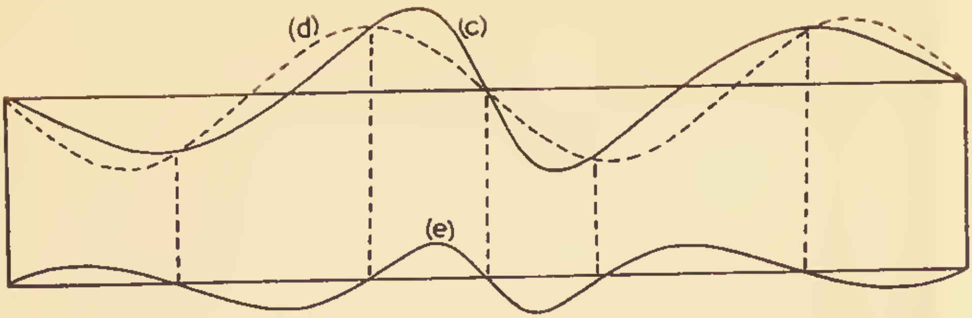

FIG. 8.2. Deduction of higher species of shallow-water tides from change of shape of progressive wave.

terrestrially generated tides, so that with a semidiurnal primary the secondary tides so generated will be of the quarter-diurnal, sixth-diurnal, and higher species of tides, while if the primary is a diurnal tide it will generate semidiurnal, third-diurnal, and higher species of tides.

8.2. Relation of shallow-water tides to primary tides

Referring again to Fig. 8.1, it is clear that a fundamental relation between curves (*a*) and (*c*) is that the two are zero together, and that they are rising together at half tide, whereas when the primary tide is falling at half tide, the shallow-water tide is rising. In the former case the two reinforce one another to produce a quick rise, while in the latter case they oppose one another so as to give a slow fall. If we represent the primary wave as proportional to

$$\cos (nt - k)$$

then the secondary wave is proportional to

$$\cos (2nt - 2k + 90°).$$

This means that if the primary wave is the tide M_2 with a phase lag k, then the shallow-water tide, called M_4, has a phase lag of

$$2k - 90° \quad . \quad . \quad . \quad . \quad . \quad (8.2a)$$

Actual analyses of tides indicate that for a very large number of places, more particularly in estuaries, this relationship is approximately according to facts.

It must not be assumed, however, that such a relationship is even approximately true for all places. In actual practice, of course, the theory is only used to indicate the type of tidal constituents to be determined by analysis, but the approximate correspondence between the results of analysis and the theoretical indications is a matter for satisfaction, and further, great departures from that which would be normally expected may serve to throw light on the character of the motions taking place. The relationship between the phase-lags of M_2 and M_4 may be used in this way.

Sufficient is known of the solution of the problem of the distortion of standing oscillations in shallow water to indicate that the phase relations are not the same as for a progressive wave, but that if the primary tide can be represented by

$$\cos (nt - k)$$

then the secondary tide can be approximately represented by

$$\cos (2nt - 2k) \text{ or by } \cos (2nt - 2k - 180°)$$

Hence, if k is the phase lag of M_2, then the phase lag of M_4 due to a standing oscillation will be approximately equal to

$$2k \quad \text{or to} \quad 2k + 180° \quad . \quad . \quad . \quad . \quad (8.2b)$$

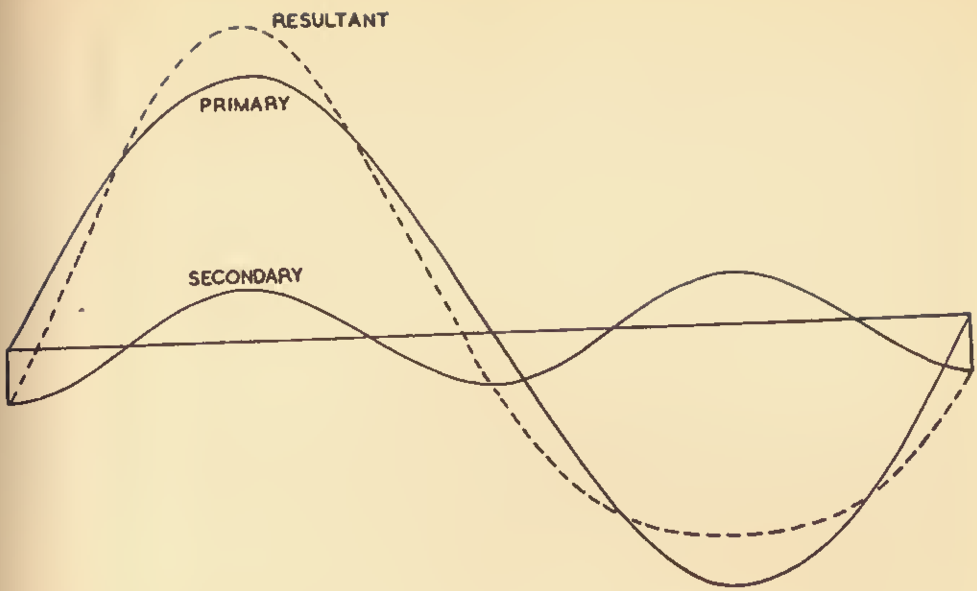

FIG. 8.3. Effects when quarter-diurnal tide is in phase with semidiurnal tide at high water.

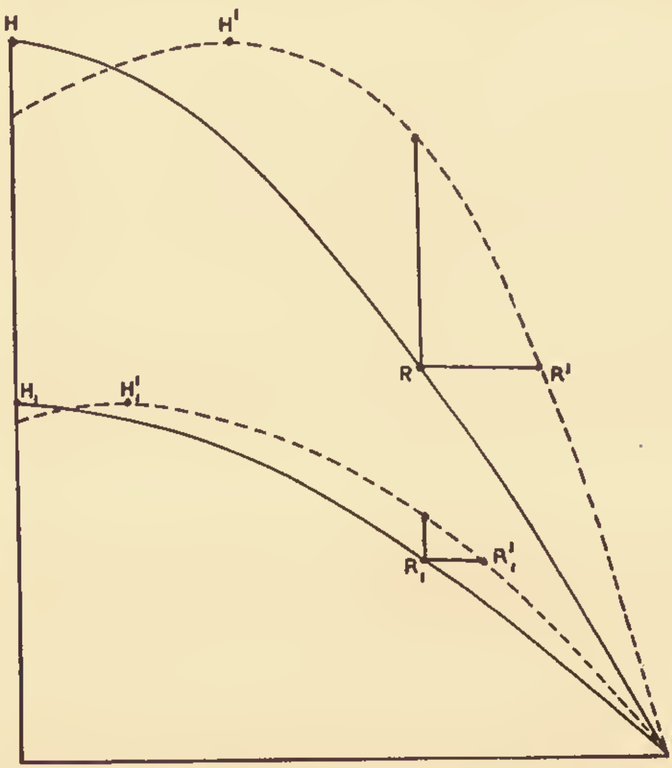

FIG. 8.4. The law of variation of range of shallow-water tides with range of semidiurnal tides.

This means that when the primary tide is at high water the secondary tide is either at high water or at low water. In the former case the high waters will have a higher and more accentuated peak, while the low waters will have a higher and flatter trough, as illustrated in Fig. 8.3.

It is of some importance to ascertain the laws governing the amplitudes of the successive species of tides as revealed by the processes of Figs. 8.1 and 8.2. Let part of Fig. 8.1 be redrawn on a bigger scale as in Fig. 8.4 and then let a similar diagram be constructed for the case where the amplitude of oscillation is halved, as in the line H_1R_1. Since the amount of distortion represented by HH′ is proportional to the actual elevation at H, it follows that the distortion H_1H_1' is only half HH′. Similarly the two points R and R_1 are at the same phase of the primary wave, so that the elevation at R is twice that at R_1, and therefore the distortion indicated by R_1R_1' is half that indicated by RR′.

Now consider the differences between the distorted and undistorted curves. It is obvious that the difference of elevations at R_1 is only about a quarter of the difference at R. This can be verified by the reader by drawing curves whose primary amplitudes are 10 and 5, and distortion such that H is moved to H′ through a space equivalent to a phase shift of 30° in the primary curve. When the amplitude is halved this equivalent phase shift is reduced to 15°. The differences between the two curves in each case are then as follows, for phases increasing by 15° in the primary curves.

TABLE 8.1

Differences in Elevation between Distorted and Undistorted Waves

Phase	Amplitude of primary oscillation		Phase	Amplitude of primary oscillation	
	10	5		10	5
0°	0·00	0·00	90°	1·34	0·17
15°	− 1·19	− 0·34	105°	2·47	0·48
30°	− 2·07	− 0·55	120°	3·07	0·68
45°	− 2·08	− 0·59	135°	3·43	0·71
60°	− 1·32	− 0·45	150°	2·41	0·59
75°	− 0·04	− 0·17	165°	1·34	0·34
90°	1·34	0·17	180°	0·00	0·00

The curves and the table show very clearly that if the amplitude of a semidiurnal tide is halved, then the amplitude of the quarter-diurnal tide will be reduced approximately in the ratio 1/4. In other words, *at any place the amplitude of the quarter-diurnal tide varies approximately as the square of the amplitude of the semi-diurnal tide.*

According to this law the quarter-diurnal tides at neaps will be less than those at springs in the ratio of the squares of the neap and spring ranges, and this is found to be generally true, or approximately so. The law is useful also in the determination of the relative importance of the harmonic constituents likely to be required to represent the shallow-water effects.

A similar law for the sixth-diurnal tide can be obtained by continuing the processes, and it is found that the sixth-diurnal tides vary in range as the *cube* of the primary semidiurnal tides, and so on.

The laws given above can be deduced directly from the mathematical equations which apply to all motion whether as a progressive wave in a channel or a stationary wave or any complex motion in a sea, so that they can be used with the utmost confidence even though the proofs here given appear to be applicable only to a particular kind of motion associated with a progressive wave. The proofs from the mathematical equations, of course, are beyond the scope of this Manual.

8.3. Shallow-water harmonic constituents

From the laws relating the shallow-water tides to the primary tide, as formulated from the theory of the progressive wave in shallow water in the previous articles, it is possible to indicate the harmonic constituents which are likely to be of most importance in connection with the analysis of tidal observations. Just as the equilibrium tide is used as a standard of reference for the harmonic tidal constituents so also is the progressive wave used as an indicator of the speeds and relative importance of the shallow-water harmonic constituents. The laws given in the preceding article apply to the tide as a whole, so that we shall proceed by expressing the tide as a number of constituents before taking tbe squares and higher powers.

Let the elevation be composed of two terms such as M_2 and S_2 and let these be denoted by $A \cos a$ and $B \cos b$ respectively. Then we write

$$y = A\cos a + B\cos b$$
$$y^2 = A^2\cos^2 a + B^2\cos^2 b + 2AB\cos a\cos b$$

This can be written as

$$y^2 = \tfrac{1}{2}A^2\cos 2a + \tfrac{1}{2}B^2\cos 2b + AB\cos(a+b) + AB\cos(a-b) + \text{constants}$$

From two harmonic terms in y we get four harmonic terms in y^2. From Table 7.1 take

$$A = 0{\cdot}908 \text{ for } M_2$$
$$B = 0{\cdot}423 \text{ for } S_2$$

then we get constituents as follows :—

	Relative Amplitude	Argument
M_4	0·41	(arg. of M_2) × 2
MS_4	0·38	(arg. of M_2) + (arg. of S_2)
S_4	0·09	(arg. of S_2) × 2
MSf	0·38	(arg. of S_2) − (arg. of M_2)

Hence three quarter-diurnal constituents have been generated, with one long-period constituent. The relative amplitudes are only of value to indicate relative importance, and strictly speaking, so far as elevations are concerned, it is only permissible to use these relative coefficients to compare constituents of the same species.

The methods available to us without resorting to complex mathematical methods are so limited that we can only *indicate* how the main terms are generated. We shall have to be content, so far as the long-period tide MSf is concerned, to show that such a tide can be generated in shallow water by the interaction of M_2 and S_2, and this will perturb the astronomically generated tide, and may even mask it completely. From now on we shall content ourselves with showing how the terms of short period, such as the quarter-diurnal, sixth-diurnal and eighth-diurnal tides, are generated, and their approximate relative importance.

Proceeding to $y^3 = y^2 \times y$ we get

$$\begin{aligned} y^3 &= \tfrac{1}{2}A^3\cos 2a\cos a + \tfrac{1}{2}AB^2\cos 2b\cos a + A^2B\cos(a+b)\cos a \\ &\quad + \tfrac{1}{2}A^2B\cos 2a\cos b + \tfrac{1}{2}B^3\cos 2b\cos b + AB^2\cos(a+b)\cos b \\ &= \tfrac{1}{4}A^3(\cos 3a + \cos a) + \tfrac{1}{4}AB^2\{\cos(2b+a) + \cos(2b-a)\} \\ &\quad + \tfrac{1}{2}A^2B\{\cos b + \cos(2a+b)\} + \tfrac{1}{4}A^2B\{\cos(2a+b) + \cos(2a-b)\} \\ &\quad + \tfrac{1}{4}B^3\{\cos 3b + \cos b\} + \tfrac{1}{2}AB^2\{\cos(a+2b) + \cos a\} \end{aligned}$$

Collecting together the terms, we get the following results :—

Argument	Coefficient	Argument	Coefficient
$3a$	$\tfrac{1}{4}A^3$	a	$\tfrac{1}{4}A^3 + \tfrac{1}{2}AB^2$
$2a+b$	$\tfrac{1}{2}A^2B + \tfrac{1}{4}A^2B$	b	$\tfrac{1}{2}A^2B + \tfrac{1}{4}B^3$
$2b+a$	$\tfrac{1}{4}AB^2 + \tfrac{1}{2}AB^2$	$2a-b$	$\tfrac{1}{4}A^2B$
$3b$	$\tfrac{1}{4}B^3$	$2b-a$	$\tfrac{1}{4}AB^2$

We now have to adopt a notation as follows :—

M_6	with argument		$= 3$ (arg. of M_2)
$2MS_6$	,,	,,	$= 2$ (arg. of M_2) + (arg. of S_2)
$2SM_6$	,,	,,	$= 2$ (arg. of S_2) + (arg. of M_2)
S_6	,,	,,	$= 3$ (arg. of S_2)
$2MS_2$	,,	,,	$= 2$ (arg. of M_2) − (arg. of S_2)
$2SM_2$	,,	,,	$= 2$ (arg. of S_2) − (arg. of M_2)

The difference between $2MS_6$ and $2MS_2$ is that the total argument in the former case is that of a sixth-diurnal constituent, and in the latter is that of a semidiurnal constituent.

We thus get, from y^3, terms as follows :—

Symbol	Relative coefficient	Symbol	Relative coefficient
M_6	0·19	M_2	0·27
$2MS_6$	0·26	S_2	0·19
$2SM_6$	0·12	$2MS_2$	0·09
S_6	0·02	$2SM_2$	0·04

It is interesting to notice that the constituents of astronomical origin, M_2 and S_2, may also be perturbed by shallow-water effects to a certain degree.

We could proceed to obtain a list of shallow-water constituents derived from y^4, but we shall refrain. Enough has been said to emphasise the importance of these constituents, and to show that one or two of each species are not sufficient ; and that M_6 is by no means sufficient to represent the sixth-diurnal constituents, though for years it was taken as the sole representative.

If we consider the constituent N_2 in addition to M_2 and S_2 we have to add other constituents such as

Symbol	Argument
MN_4	(arg. of M_2) + (arg. of N_2)
$2MN_6$	2 (arg. of M_2) + (arg. of N_2)

and there are others such as

MK_4 and $2MK_6$.

The diurnal constituents are not usually so great as the semidiurnal constituents, but there are interactions with M_2 and S_2. Following the analogy of the investigation given above, we see that the required terms are those with arguments obtained by adding or subtracting the arguments of the primary terms, so that we get

Symbol	Argument
MK_3	(arg. of M_2) + (arg. of K_1)
MO_3	(arg. of M_2) + (arg. of O_1)

(The latter is sometimes called $2MK_3$ on the curious assumption that it is derived from an interaction of M_4 and K_1.)

For the values of f and u we have a rule which is illustrated for one case:

If argument is	i (arg. of A)	+ (arg. of B)
then u is	i (u of A)	+ (u of B)
and f is	(f of A)^i	× (f of B)

where A, B are any two constituents involved, and i is an integer. Thus

for M_4 we have $u = 2$ (u of M_2), $f =$ (f of M_2)^2
and for MS_4 we have $u = u$ of M_2, $f = f$ of M_2

8.4. The importance of shallow-water constituents

The distortional effects caused by shallow water are so great in many estuaries that the number of shallow-water constituents becomes unmanageable. In fact, the processes similar to those outlined in Art. 8.1 show that the successive species of tides (quarter-diurnal, sixth-diurnal . . .) do not diminish in importance so rapidly as was once thought. The " convergence " is said to be slow.

It is in any case futile to take only one term of a species. For a simple illustration suppose M_2 has an amplitude of 10 ft. and S_2 has an amplitude of 5 ft., and only M_6 has been evaluated with an amplitude of 0·2 ft. Now the laws of powers of the range of tide (Art. 8.2) indicate that if for the average amplitude of tide (10 ft.) the sixth-diurnal tide has an amplitude of 0·2 ft., then at spring tides in this instance it will have an amplitude of $(15/10)^3 \times 0{\cdot}2$ ft. $= 0{\cdot}7$ ft., and at neap tides it will have an amplitude of $(5/10)^3 \times 0{\cdot}2$ ft. $= 0{\cdot}03$ ft. Clearly, therefore, a single term is very inadequate to express the effect.

Special methods of a highly technical nature have had to be evolved to cope with such a problem as this, in connection with the prediction of tides (see Chapter XV).

CHAPTER IX

THE ADMIRALTY METHOD

9.1. The relation of the Admiralty method to the equilibrium forms

THOUSANDS of tidal analyses have been made for places scattered over the earth, and experience shows that generally (but not always) the equilibrium relations are to a certain extent exemplified in the actual constants found by analysis. The question therefore arises—How far is it possible to use the equilibrium expressions? The difficulties associated with the direct applications of the equilibrium expressions led to the development of the harmonic method, particularly for the analysis and prediction of mixed tides, in which the diurnal and semidiurnal tides are of comparable importance. Within the past few years, however, it has been found possible to overcome some of the difficulties associated with the equilibrium forms, and a practicable method (the Admiralty method as it is called) has been evolved, suitable for the computation of approximate predictions. Whereas the older methods of approximate predictions used only the approximate relation between the time of high water of the semidiurnal tide and the lunar transit, and thus were only applicable where the semidiurnal tide was predominant, the new method is directly related to the equilibrium forms and makes adequate allowance for the changes of parallax and declination, and also 19-yearly inequalities in declination, which were ignored in the older methods. Thus the Admiralty method is the logical and practical outcome of the equilibrium forms, but its derivation and justification are largely based upon the harmonic method.

9.2. The variations in equilibrium relations

The equilibrium relations between the tidal constituents are *not* fulfilled by the tides as a whole because the diurnal and semidiurnal tides exhibit very marked differences in their responses to the generating forces, so much so that there is no possibility that a method can be devised for the expression of tides which does not demand separate treatment of the diurnal and semidiurnal tides. Again, it is found that while the constituents M_2 and S_2, for instance, are related to one another in the equilibrium tide in such a way that their amplitudes are in the ratio $1 : 0{\cdot}460$, in nature this ratio varies considerably, and under peculiar circumstances S_2 may be even greater than M_2. Taking average conditions, it may be said that the ratio of amplitudes of S_2 and M_2 is approximately the same as the equilibrium ratio. It is found, however, that in most places (but not in all) the actual tide lags behind the equilibrium tide by a greater amount in the case of S_2 than in the case of M_2, the average difference of the phase-lags being about 30°.

The response of the water, as was pointed out in Chapter IV, will vary with the speed, and we should expect therefore that the ratios of amplitudes of actual constituents to equilibrium constituents at any one place will vary approximately according to the speed, so that if, for instance, the ratio of amplitude of S_2 and M_2 is smaller than the equilibrium ratio, then the ratio of amplitude of N_2 and M_2 will be greater than the corresponding equilibrium ratio, because the speeds of S_2 and N_2 are on opposite sides of the speed of M_2. Again, because the speeds of N_2 and M_2 are more nearly equal than those of S_2 and M_2, we should expect N_2 and M_2 to have a closer relationship and correspondence with equilibrium constituents than we should expect to have in the case of S_2 and M_2. Further, because N_2 is smaller than S_2 (as a rule), it follows that the error in assuming N_2 and M_2 to be related together according to equilibrium conditions is much less than would result from a similar assumption regarding S_2 and M_2.

9.3. Brief description of the Admiralty method

The Admiralty method uses harmonic constants g and H for the principal constituents M_2, S_2, K_1 and O_1, as obtained from Part II of the Admiralty Tide Tables. Thus it does not attempt to modify the equilibrium tide as a whole, nor does it assume that the equilibrium relations apply to the semidiurnal tide as a whole, or to the diurnal tide as a whole. Reasons will be given in the ensuing discussions to show that there are satisfactory grounds for applying certain equilibrium relations to these four principal constituents.

Thus, the first object of the Admiralty method is to modify the harmonic constants g and H by certain equilibrium relations so that

$$g \text{ is replaced by } g + b + c,$$
$$\text{H is replaced by BCH}$$

where b, c, B, C are specially defined, as below.

These amendments are readily effected by the use of tables for the day of the year, whether in permanent forms, as in Part III of the Admiralty Tide Tables, or for the day of the current year, as in Part I of the Admiralty Tide Tables. The modified constituent is thus written as

$$\text{BCH} \cos (nt - g - b - c)$$

In this formula

n = the speed
B = a factor } depending upon the year and the day of the year.
b = an angle }
C = a factor depending upon the moon's parallax.
c = an angle depending upon the longitudes of the moon and sun, or in practice, less accurately, upon the time of the moon's transit.

Thus S_2 is corrected for K_2 and T_2 and nodal variations in K_2.
M_2 is corrected for nodal variations, and also for N_2 and other lunar constituents.
K_1 is corrected for P_1 and nodal variations in K_1.
O_1 is corrected for nodal variations and also for other lunar constituents.

The next process is to combine the two modified semidiurnal constituents into a compound one, and the two modified diurnal constituents into a compound one. A table is provided for this purpose, but there is an assumption that for small values of t we can ignore the variations in the speed of the constituents of the same species, and it is found convenient to regard the compound constituents as having solar speeds, 30° and 15° respectively.

By the aid of another table, relative amplitudes and phase differences of the compound semidiurnal constituent and the compound diurnal constituent are used to give the changes in the high water and low water times and heights of the semidiurnal tide, due to the diurnal tide, and then the approximate times and heights of the high and low waters are obtained.

More accurate times of high and low water can be computed, whenever special accuracy is desired, by making use of a table of corrections which allow for the variations in the periods of the semidiurnal and diurnal tides, and both times and heights may be corrected, if necessary, by applying certain "shallow-water corrections."

The process is very simple, as may be seen from the example in Table 9.1, extracted from the Admiralty Tide Tables, Part III, to which reference should be made for a fuller description of the use of the method and its variations.

The method, though it deals adequately with the very difficult problems associated with the prediction of mixed diurnal and semidiurnal tides, involves very little computation. When familiarity with the process has been attained, the computations can be effected with ease in a few minutes, and the results are remarkably good.

Table 9.1

Form A.
To find times and heights of high and low water.

Example 1. To use form as published: $b + c$ and B × C being extracted from Table of Tidal Angles and Factors, A.T.T., Part I.

M_0 or Z_0 6·6 ft.
Seasonal Corr. + 0·5 ft.
Corr. M_0 or Z_0 7·1 ft.

Required to find times and heights of High and Low Water at Thursday Island on 21st February, 1941.

		M_2	S_2	K_1	O_1
Extracts from A.T.T., Pt. II. For the place extract M_0 or Z_0 and compute the seasonal correction for date (when given in Table 2). Enter at head of form. Enter g and H for the four constituents.	(1)	g 066° H 1·2 ft.	g 336° H 1·1 ft.	g 205° H 1·9 ft.	g 146° H 1·0 ft.
Extracts from A.T.T., Pt. I. Enter $b + c$ and B × C from Table of Tidal Angles and Factors.	(2)	$b+c$ 253° B×C 1·06	$b+c$ 010° B×C 1·17	$b+c$ 321° B×C 0·77	$b+c$ 315° B×C 0·83
Not used for these particular predictions.	(3)	t' ° — —	— — — —	— — — —	t' ° — —
Add $g + (b + c)$ to give m, s, k, o. Multiply H × (B × C) to give M, S, K, O.	(4)	m 319° M 1·3 ft.	s 346° S 1·3 ft.	k 526° K 1·5 ft.	o 461° O 0·8 ft.
Combination of constituents into semidiurnal and diurnal tides.					
Compute d, D as instructed, add or subtract 360° or 720° if necessary.	(5)	$d_2 = m - s$; $D_2 = \frac{M}{S}$	d_2 333° D_2 1·0	d_1 295° D_1 0·5	$d_1 = o - k$; $D_1 = \frac{O}{K}$
Enter Table 3, A.T.T., Pt. III., with d, D from line (5). One decimal will suffice for E.	(6)		e_2 347° E_2 1·9	e_1 340° E_1 1·3	
Compute f, F as instructed.	(7)	$f_2 = e_2 + s$; $F_2 = E_2 \times S$	f_2 693° F_2 2·5 ft.	f_1 866° F_1 2·0 ft.	$f_1 = e_1 + k$; $F_1 = E_1 \times K$
Subtract 360° or 720° if necessary so that f_2 and f_1 are less than 360°.	(8)		f_2 333° —	f_1 146° —	
Convert f_2 and f_1 angles into time: $\frac{f_2}{30}, \frac{f_1}{15}$.	(9)		f_2 11·1 hrs. —	f_1 9·7 hrs. —	

COMBINED SEMIDIURNAL AND DIURNAL TIDE

To combine semidiurnal and diurnal tide.							
Compute j and J as instructed.	(10)	$j = (f_1 - f_2)$ hrs.; add 24 hrs. if negative $J = \frac{F_1}{F_2}$	j 22·6 hrs. J 0·8				
	(11)	Enter Table 4, A.T.T., Pt. III, with j and J, interpolating if necessary.	Time (l) Factor (L)	6·7 hrs. − 1·4	12·4 hrs. 0·3	17·2 hrs. − 0·8	23·8 hrs. 1·8
Compute q from (9) and (11). Compute Q from (7) and 11).	(12)	$q = (l + f_2)$ hrs. $Q = (L \times F_2)$ ft.	Time (q) Height (Q)	17·8 hrs. − 3·5 ft.	23·5 hrs. 0·8 ft.	28·3 hrs. − 2·0 ft.	34·9 hrs. 4·5 ft.
†TIMES AND HEIGHTS OF HIGH AND LOW WATER.	(13)	*Convert q to hrs. and mins. (Subtract 24 when q is greater than 24 hrs.) Add Q and corrected M_0 or Z_0.	Time Height	17 h. 48 m. 3·6 ft.	23 h. 30 m. 7·9 ft.	4 h. 18 m. 5·1 ft.	10 h. 54 m. 11·6 ft.
For special accuracy. Use Table 5 and compute corrections from formula at foot of table.	(14)	Enter Table 5, A.T.T., Pt. III, with d_2, D_2 for S_2 corrections and d_1, D_1 for K_1 corrections from (5).	Corrections Corrd. times	+ 6 m. 17 h. 54 m.	+ 12 m. 23 h. 42 m.	− 8 m. 4 h. 10 m.	− 1 m. 10 h. 53 m.

Application of shallow water corrections. Enter times from (14) heights from (13), corrections from Table 1, A.T.T., Pt. II.	(15)	h. m.	ft.	h. m. No	ft. SHALLOW WATER CORRECTIONS	h. m.	ft.	h. m.	ft.
CORRECTED PREDICTIONS, times from (14), heights from (13).	(16)	04 h. 10 m.	5·1 ft.	10 h. 53 m.	11·6 ft.	17 h. 54 m.	3·6 ft.	23 h. 42 m.	7·9 ft.

* Tides from which 24 hours have been subtracted become first tides of the day. The subsequent application of shallow water corrections may cause a tide to occur earlier than 0000 or later than 2400. Such tides are more accurately computed by using the astronomical data appropriate to the day in which they now occur.

† At places where there are no shallow water corrections, these times and heights will give a good approximation of all tides, and will be correct for a tide occurring at 1200.

9.4. The solar semidiurnal tide

The principles of Art. 9.2 are illustrated in a very striking manner by the solar semidiurnal tide, because the constituents of this tide have speeds very nearly equal to one another. In practically all analyses the results show that we can assume the equilibrium relations between K_2 and S_2 so that their amplitudes are in the equilibrium ratio, and they have the same phase-lag (almost exactly) on the equilibrium constituents. This is all the more remarkable because K_2 is partly of lunar origin and partly of solar origin. Hence in seeking an approximate method we can at once assume equilibrium relations and consider K_2 and also T_2 as constituents which give a variation of S_2. Further, as both K_2 and T_2 have arguments depending principally upon the mean longitude of the sun, and therefore upon the day of the year, we have a compound constituent S_2 whose amplitude and phase vary with the date. There is, however, another variation in K_2 which depends upon the revolution of the moon's nodes, for it is of declinational origin. This nodal variation, of course, means that the compound S_2 needs to be modified slightly according to the year in a cycle of 18·61 years.

9.5. The lunar semidiurnal tide

In the case of N_2, the reasons already given appear to be sufficient invariably for assuming the equilibrium relations with M_2 as a good *approximation*, but only as an approximation. Returning to the equilibrium expressions, we found from Art. 6.2 that N_2 is derived from the parallax factor in conjunction with the movement of the moon in longitude (*i.e.*, in the ecliptic), and that the equilibrium relations, if adopted, would also bring in ν_2, μ_2, L_2 and $2N_2$. Hence if we took the argument of M_2 as involving twice the mean longitude of the sun less twice the true longitude of the moon (*i.e.*, $2h - 2x$) instead of the mean values $(2h - 2s)$, and if we used the parallax factor we should get a good allowance for N_2 and four other important constituents.

Since the constituents M_2, N_2, . . . have the same values for the nodal variations f, u, it follows that the modified constituent can be treated like M_2. Hence, by taking true lunar longitudes in place of mean lunar longitudes for M_2, we allow for a number of important constituents. We have, in effect, assumed that such a constituent as N_2 can be related to M_2 according to the equilibrium relations, and in measure this is true enough for the ratio of amplitudes, but there is a source of error in connection with the difference in phase-lags. It is well known that compared with the corresponding equilibrium constituents the constituent S_2 has a greater lag than M_2, and similarly N_2 has a smaller lag. On the average over the world the lags of N_2 and M_2 differ by about 15°. The error due to this source, for a difference of 15°, may amount in height of high water to about 25% of the amplitude of N_2 (or 5% of the amplitude of M_2) and in high water time to about six minutes. While there is not a great degree of variability in the differences of lags of N_2 and M_2, apart from very exceptional cases, it is not easy to allow for it, and corrections may be of doubtful value.

9.6. Use of transits for lunar semidiurnal tide

It will be noted that in the above remarks we have given the argument of the modified M_2 as involving the true longitude. Now it is customary in Nautical Almanacs to give transits of the moon, and as lunar longitudes are not in common use, the transits may be used for the sake of convenience, with some slight loss of accuracy. The true transit depends upon the difference between the right ascensions of the mean sun and the true moon. The equilibrium relations would immediately point to the use of transits, because the angle Z is intimately related to the right ascension of the moon (5.3f), but the difference between the longitude and the right ascension involves small terms which are partly utilised in the lunar part of K_2 and partly also in the nodal corrections, f, u for M_2.

We could, of course, make allowance for this, by introducing values of K_2, f, u, which would pertain to the Admiralty method when using transits, but the confusion

which would ultimately result from this artifice would be so very great that it is better simply to recognise the fact that the values of K_2, f, u, used in tbe Admiralty method are only partly correct when transits are used. The errors so caused may amount to as much as 5% of the amplitude of M_2 in height and 6 minutes in time, though, of course, the average error will be small.

If more accurate methods are desired it is a simple matter to use the longitudes instead of the transit-angle, and in fact, the annual tables of "Tidal Angles and Factors," published in Admiralty Tide Tables, Part I, utilise lunar longitudes in place of transits. When these are available they should be used for the sake of accuracy as well as convenience.

9.7. The diurnal tide

In the case of the diurnal tides, similar principles can be utilised. The constituents K_1 and P_1 are intimately related at all places according to the equilibrium relations, because their speeds are very nearly equal. The constituent O_1 has its argument depending on $h - 2s$, and if we replace the mean longitude s by the true longitude x and apply the parallax factor, we immediately get a compound constituent made up of O_1, Q_1, and part of M_1. Similarly, the parallax factor applied to the lunar part of K_1 (*i.e.*, a smaller parallax factor applied to the whole of the luni-solar K_1) will also include J_1 and the remaining part of M_1.

The argument of the modified O_1 may now be written in the form

$$-h + (2h - 2x)$$

of which the latter part is the modified equilibrium argument of M_2, and the former part is related to the day of the year. This is an important relation because it enables us to simplify the use of extractions of astronomical data. The argument of O_1 can be partly tabulated in terms of date, with nodal corrections for the nodal period of 18·61 years, and the remaining part can be expressed in terms of the modified equilibrium argument of M_2.

If transits are used to determine the modified equilibrium argument of M_2 the results can be used as part of the argument of O_1, and it is found that the error involved in using the right ascension of the moon in place of the longitude is entirely negligible for the diurnal tide.

9.8. General remarks

It is well to note that for well over a century it was customary to obtain approximate predictions for a given place by adding a fixed constant, appropriate to the place, to the times of transits of the moon, and if more accurate approximations were required some allowance was made for the solar tide S_2 by assuming an invariable relationship with the lunar tide M_2. The diurnal tide was entirely ignored. In the Admiralty method, not only is the diurnal tide allowed for, but the semidiurnal tide is much more accurately expressed, so that S_2 is given its proper value, and further, many other important constituents are utilised. While it is desirable to point out the imperfections necessarily existing in the method, these should be considered in relation to the great gain that has been made.

*9.9. Errors in use of transits instead of longitudes

The remarks made in Art. 9.6 above concerning the use of right ascension can be verified from Table 7.5. The constituent K_2 is partly indicated by the term with argument $30°z$. When we substitute for the angle a in terms of s, the term with angle $(30°z - 4a)$ is cancelled and the coefficient of the term with argument $30°z$ is doubled, as may be seen from Table 7.9. Similarly in the case of the nodal terms, in Table 7.5 there are two of equal size with arguments $(30°z - 2a + N)$ and $(30°z - 2a - N)$ and when a is replaced by s the former is doubled and the latter is cancelled. The error in height is clearly due to a constituent of relative amplitude 0·044 (*i.e.*, 0·040/0·914 times the amplitude of M_2) and the error in time is due to a constituent of the same size.

* See par. 1, page vii.

*9.10. Note on the computations of certain tables

The Admiralty method depends for its computational possibility upon two tables:

(1) Table 3, for combining constituents of the same species (semidiurnal or diurnal);

(2) Table 4, for combining tides of different species (semidiurnal and diurnal) and it is desirable to place on record the mode of computation of these tables.

The first one is simply computed, for if we have two constituents

$$M\cos(nt - m) + S\cos(nt - s) \qquad . \quad . \quad . \ (9.10a)$$

where each is taken as having the same speed, then we can reduce these by a change of time origin and amplitudes to give

$$S\{\cos nt' + D\cos(nt' - d)\} \qquad . \quad . \quad . \quad . \ (9.10b)$$

where

$$\left.\begin{aligned} nt' &= nt - s \\ D &= M/S \\ d &= m - s \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \ (9.10c)$$

We can write (9.10b) in the form.

$$S.\ E\cos(nt' - e) \qquad . \quad . \quad . \quad . \quad . \ (9.10d)$$

with

$$\begin{aligned} E\cos e &= 1 + D\cos d \\ E\sin e &= D\sin d \end{aligned} \qquad . \quad . \quad . \quad . \ (9.10e)$$

and the table provided is based on the formulæ last given, by taking tabular values of D and d and computing E and e according to the formula.

For the table which is used to combine tides of different species, we write the two constituents (semidiurnal and diurnal) in the form

$$F_2\cos 30^\circ(t - f_2) + F_1\cos 15^\circ(t - f_1) \quad . \quad . \quad . \ (9.10f)$$

which is equal to

$$F_2\{\cos 30^\circ t' + J\cos 15^\circ(t' - j)\} \quad . \quad . \quad . \quad . \ (9.10g)$$

where

$$\left.\begin{aligned} t' &= t - f_2 \\ J &= F_1/F_2 \\ j &= f_1 - f_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \ (9.10h)$$

Table 4 given with the Admiralty method is computed from

$$\cos 30^\circ t' + J\cos 15^\circ(t' - j) \quad . \quad . \quad . \quad . \ (9.10i)$$

It can be shown mathematically, or verified by computation, that for a mixed oscillation represented by this formula the times of high and low water must be given at the time t', satisfying

$$2\sin 30^\circ t' + J\sin 15^\circ(t' - j) = 0 \quad . \quad . \quad . \ (9.10j)$$

or

$$\sin 15^\circ(t' - j) = -\frac{2\sin 30^\circ t'}{J} \qquad . \quad . \quad . \ (9.10k)$$

By taking $15^\circ t'$ at intervals of one degree, and by taking specified values of J, the values $-(\sin 30^\circ t')/J$ were computed and these gave $15^\circ(t' - j)$ whence the values of j were obtained by subtraction from the values of $15^\circ t'$.

Having obtained the value of j in terms of J and t' it only remained to interpolate so as to give t' in terms of J and j, and Table 4 of the Admiralty method gives the various values of t' appropriate to the high and low waters.

The factors for the heights of high and low water were obtained by substituting the values of t' in (9.10i).

* See par. 1, page vii.

***9.11. Derivation of quantities B, C, b and c**

The general expression for a harmonic constituent (see Art. 6.7) is

$$fH \cos (V + u - \kappa)$$

where f and u are nodal quantities, V is the astronomical argument given in the table of Art. 7.1, and κ is the phase lag. If we take the astronomical arguments as for Greenwich we replace κ by g, as defined in Art. 7.3.

Now in the case of the principal lunar semidiurnal tide, the value of V is $30°t + 2h - 2s$, where h and s are the mean longitudes of the sun and moon respectively and t is the time in mean solar hours measured from mean solar midnight. In the Admiralty method, as described in Art. 9.5, we introduce a parallax factor C and replace s by the true longitude of the moon (x), in order to include the effects of N_2 and other constituents, and so we get the lunar semidiurnal tide

$$fCH \cos (30°t + 2h - 2x + u - g) \qquad . \quad . \quad . \quad (9.11a)$$

Here $$C = \left(\frac{\text{the horizontal parallax of the moon}}{\text{the mean horizontal parallax of the moon}}\right)^3 = \left(\frac{c}{r}\right)^3 \qquad . \quad . \quad (9.11b)$$

and in the general notation of Art. 9.3 we have

For the modified M_2 $$\left.\begin{cases} B = f \text{ of } M_2 \\ b = -u \text{ of } M_2 \\ c = 2x - 2h \end{cases}\right\} \qquad . \quad . \quad . \quad . \quad (9.11c)$$

In the more accurate form of the method c is taken as here defined, but in the less accurate form which conveniently uses transits then

$$c = (\text{time of transit in hours}) \times 29° \qquad . \quad . \quad . \quad (9.11d)$$

The diurnal lunar constituent O_1 is expressed in the general notation for a harmonic constituent as

$$fH \cos (15°t + h - 2s - 90° + u - g)$$

which can be written as

$$fH \cos (15°t + 2h - 2s + u - h - 90° - g)$$

When we include the parallax factor C, and replace s by x as in the case of M_2 then in the notation of Art. 9.3.

For the modified O_1 $$\left.\begin{cases} B = f \text{ of } O_1 \\ C = \text{parallax factor given in (9.11b)} \\ b = (h + 90°) - (u \text{ of } O_1) \\ c = 2x - 2h \end{cases}\right\} \qquad . \quad . \quad (9.11e)$$

The same remarks apply to the term c as in the case of the modified constituent M_2.

The tabulation of B and b is obviously very simple in the cases of M_2 and O_1, but in the cases of S_2 and K_1 the computations are complicated by the compounding of constituents. We shall take the diurnal constituent first. In the general harmonic notation of Art. 7.1 we have

$$\left.\begin{aligned} K_1 &= fH \cos (15°t + h + 90° + u - g) \\ P_1 &= 0{\cdot}331 \text{ H} \cos (15°t - h - 90° - g) \end{aligned}\right\} \qquad . \quad . \quad (9.11f)$$

where f, H, u and g pertain to K_1, and g can be taken as the same for both constituents, according to Art. 9.7. The factor 0·331 is the equilibrium ratio of amplitudes of P_1 and K_1. The compound tide is written and used in the form

$$BH \cos (15°t - g - b) \qquad . \quad . \quad . \quad . \quad (9.11g)$$

* See par. 1, page vii.

For the purpose of computation we find it simpler to write this temporarily as

$$\text{BH} \cos\{(15^\circ t + h + 90^\circ - g) - (b + h + 90^\circ)\} \qquad . \quad . \ (9.11\text{h})$$

whence, on expanding and comparing with the sum of the terms for K_1 and P_1, we get

$$\left.\begin{array}{l} \text{B} \cos(b + h + 90^\circ) = \quad f \cos u + 0{\cdot}331 \cos 2\,(h + 90^\circ) \\ \text{B} \sin (b + h + 90^\circ) = - f \sin u + 0{\cdot}331 \sin 2\,(h + 90^\circ) \end{array}\right\} \quad . \ (9.11\text{i})$$

The method of computation is now simple; values of $(h + 90^\circ)$ for each day of the year are computed and can be taken the same for each year, and so can the tables of $0{\cdot}331 \cos 2\,(h + 90^\circ)$ and $0{\cdot}331 \sin 2\,(h + 90^\circ)$. The nodal terms $f \cos u$ and $f \sin u$ vary in a period of 18·61 years, but are readily incorporated in (9.11i). Thence values of B, $(b + h + 90^\circ)$, and b are easily computed.

In the case of S_2 the computations are a little more complicated still, for we have to include K_2 and T_2. We can write from Art. 7.1, and Art. 7.2 in relation to the argument of T_2,

$$\left.\begin{array}{l} S_2 = \text{H} \cos (30^\circ t - g) \\ K_2 = 0{\cdot}272 f\text{H} \cos (30^\circ t + 2h + u - g) \\ T_2 = 0{\cdot}059\ \text{H} \cos (30^\circ t - h + 282^\circ - g) \end{array}\right\} \quad . \quad . \ (9.11\text{j})$$

where H and g pertain to S_2, f and u to K_2 and the factors 0·272 and 0·059 are the equilibrium ratios of K_2 and T_2 to S_2, in accordance with Art. 9.4. The modified constituent S_2 is written

$$\text{BH} \cos (30^\circ t - g - b) \qquad . \quad . \quad . \quad . \ (9.11\text{k})$$

and therefore

$$\left.\begin{array}{l} \text{B} \cos b = 1 + 0{\cdot}272 f \cos (2h + u) + 0{\cdot}059 \cos (h - 282^\circ) \\ \text{B} \sin b = \quad - 0{\cdot}272 f \sin (2h + u) + 0{\cdot}059 \sin (h - 282^\circ) \end{array}\right\} \quad . \ (9.11\text{l})$$

Alternatively, for ease of computation, we can write

$$\left.\begin{array}{l} \text{B} \cos (b + 2h) = \cos 2h + 0{\cdot}059 \cos (3h - 282^\circ) + 0{\cdot}272 f \cos u \\ \text{B} \sin (b + 2h) = \sin 2h + 0{\cdot}059 \sin (3h - 282^\circ) - 0{\cdot}272 f \sin u \end{array}\right\} \quad . \ (9.11\text{m})$$

whence B, $(b + 2h)$, and b are readily derived.

The method of prediction described in this chapter has now been superseded by the method described on Admiralty Prediction Form H.D. 289.

CHAPTER X

TIDE GAUGES AND CURRENT METERS

10.1. The tide staff

ANY apparatus for measuring the height of the tide is a tide gauge, and the most elementary method of observing tides is by eye observations of the level of the water on a graduated staff or plank. For very many years this was the only type of tide gauge used, and no dock was, or is, complete without such a staff attached to the outer wall of the entrance; alternatively the graduations may be engraved on the wall itself. The staff must of course be vertical; if it cannot be so fixed, or if the wall to be engraved is not vertical, the graduations must be lengthened according to the slope, so that tidal movement may be shown correctly. Reference to the staff is sufficient to show the heigbt of the tide and to ensure the opening and closing of the dock gates according to the tidal movement. Any graduations may be used, and the staff set up so as to record water movements with reference to any convenient local level, usually the dock sill.

Provided the staff is long enough to permit of the extreme values of the tidal elevation being recorded, all the tidal constituents may be evaluated from a long-continued series of observations of the staff, though the actual heights will have no meaning except to those habitually using the staff. By direct levelling, however, surveyors can relate the zero of the graduations on the staff to fixed marks, the levels of which are known with reference to the zero of the national survey (Ordnance datum in Great Britain). This matter will be discussed in greater detail in a later chapter, but it is necessary here to emphasise that the readings on the tide staffs or scales commonly seen at the entrances to docks have no universal meaning.

A properly graduated and well-fixed staff is invaluable in tidal work; indeed it is the fundamental tide gauge governing in most instances the settings of automatic tide gauges. If carefully maintained it is permanent and free from all errors, apart from those due to the difficulty of obtaining exact eye observations of the height of the sea surface on the staff.

Tide staffs are also used for more temporary needs, in the surveys of coasts and by expeditions. In the former case exact knowledge of the relation between the zero of the staff and the land datum is essential to the use of the staff and it is desirable in every case that the zero of the staff should be related to fixed marks on shore. The staff must always be vertical, and in order to diminish errors in the readings due to waves on the sea surface it should be set up in a sheltered place.

10.2. Tide indicators

Instead of a fixed tide staff, a floating pole may be used, so placed that tbe graduations may be read against a fixed mark. If tbe pole is floating in a fixed cylindrical shaft or well to which the sea is admitted only at a considerable distance below the surface, wave action is considerably diminished. If the orifice by which the sea is admitted is also small, wave action may be eliminated almost entirely, but care must be taken that the orifice is not too small or the tidal motion itself will be decreased in the shaft (see Art. 10.4).

Simple devices of this nature bave been used as tide indicators, by gearing attached to the floating pole so as to operate a finger on a large dial, or other method. If such apparatus is made on a sufficiently large scale, and illuminated at night, it may be set up in an estuary or other place where knowledge of the height of the tide is important to navigation, and used to indicate the height to passing vessels. The zero reading of the dial must of course be related to the datum of the chart if the readings are to be of use to strangers.

It may be remarked that such a device is in the direct line from the simple aid to navigation of former days, that the depth of water in a channel was sufficient when a certain sandbank or rock in its approach was awash or covered by the tide.

10.3. Tide wells

The use of an enclosed shaft or well has many advantages over the tide staff exposed to all the wave action of the sea surface. Accurate readings of the tide staff can only be made in calm weather and by experienced observers, even when it is set up in a sheltered position. It is impossible to read a tide staff if it is entirely enclosed, so that a staff cannot be erected in a well which is shut off from wave action. Either a floating pole must be used, or some device must be adopted for obtaining readings of the rise and fall of the water surface in the well with reference to a fixed mark. Several such devices have been used.

It is desirable that, where a continuous record of the tide is to be obtained, the observing station should be sheltered from the weather ; a floating tide pole needs to be very long for its movement is the whole range of the tide ; consequently a lofty roof is required to the shelter over it. A flexible wire may, however, be attached to a body floating in the well and passed over a wheel, which is rotated as the tide rises and falls. The wire is kept taut by a counterpoise. The wheel may be connected by suitable gearing to a smaller wheel, which indicates directly the height of the tide. This principle, of course, can be used to operate a tide indicator. It should be noted, however, that if a floating pole is not used, the device must be calibrated against a tide staff outside the well, and thus some of the advantages of the well are lost. Further, any flexible connection introduces the risk of error due to slipping of the wire on the wheel. We shall deal later with the principles used in transmitting motion from a float to a recorder.

Another method of observing the rise and fall of the water in a well is to obtain measurements of the depth of water in the well, by means of a metal tape, which is let down till the plummet makes contact with the water. The tape is then read against a fixed mark at the top of the well, and this measurement subtracted from the length of the well from the mark to the bottom gives the depth of water in the well. Great accuracy cannot be obtained by eye, for the instant of contact between the plummet and the water is doubtful ; a better method is to use an electrical indication of the contact. Sea water is a conductor of electricity, so all that is required is a battery (say a small dry cell such as is in common use) in circuit with the tape, a small voltmeter, and a metal connection in the sea. When the plummet touches the sea surface the voltmeter is instantly operated.

There are many variations of this electrical method, which is now standard at primary tidal stations (for calibrating and checking the recording instruments, not for obtaining tidal observations).

Whatever the method of observing adopted, the fixed mark of reference should be related to the standard datums of the surveys of the locality, so that the mere addition or subtraction of a constant from the readings gives the heights of the tide with reference to those datums.

10.4. Dimensions of the well and orifice

An automatic tide gauge may be set up on land, near the sea and not much above high water mark, or on a pier or other structure standing in the sea. In the first case, the well must be excavated and connected with the sea by a pipe below the lowest level to which the tide falls. In the second case, a well may be contrived under the structure, carried down to a level below the lowest low water, and connected with the sea by an orifice below the lowest low water level ; except for this orifice, the well should be water tight. In either case, a cock should be provided by means of which the sea connection can be closed at high water and the connection flushed out by opening it at low water. The orifice should be a little above the bottom of the well, so that the lower part forms a trap for silt, which may be cleared out from time to time. The loss of a few hours' tidal record is of

little importance, but the flushing of the sea connection and clearing away of silt is vital to the correct working of the gauge.

If the well is to be free from wave action at all states of the tide the orifice must be below the lowest low water ; if the water is very shallow at the place where the gauge is erected it may be advantageous to carry a pipe out into deep water, but such a pipe will easily be choked by weed. It is well known that the effects of surface disturbances fall off very quickly with the depth, and in deep water at a depth from the surface equal to half the wave length of the surface wave the range of oscillation of the water particles is reduced to 1/23rd of that at the surface. In shallow water the water at the bottom is still less disturbed by the passage of surface disturbances.

The size of the orifice is of importance, for it is essential that water level in the well shall agree exactly with sea level at all times. A well-known theorem called Torricelli's theorem asserts that the velocity of water in an orifice is given by

$$\sqrt{2gh} \qquad \text{(10.4a)}$$

where h is the " head " of water (*i.e.*, the difference between the levels of the water on either side of the orifice) and g is the usual coefficient of gravitational force. It is customary to allow in the theory of hydraulics for frictional and other forces so that we can take the approximation

$$0{\cdot}6\sqrt{2gh} \qquad \text{(10.4b)}$$

for the velocity of the water through the orifice. If the area of the well is denoted by A and that of the orifice by a, then it follows that

$$\text{rate of rise of water in the well} = \frac{a}{\text{A}}0{\cdot}6\sqrt{2gh} \qquad \text{(10.4c)}$$

If we measure the rates in feet per second and the difference in elevation in feet, the formula becomes approximately

$$\text{rate of rise in well in feet per second} = 5\frac{a}{\text{A}}\sqrt{\text{difference in feet}} \qquad \text{(10.4d)}$$

Suppose for example that a circular well has a diameter ten times greater than that of the circular orifice and that, at a particular moment, sea level is 0·1 foot higher than water level in the well, then we have

$$\begin{aligned}\text{rate of rise in the well} &= 5 \times 1/100 \times \sqrt{0{\cdot}1}\\ &= 0{\cdot}015 \text{ foot per second.}\end{aligned}$$

At this rate it would take 7 seconds to reach equilibrium, but as the rate of rise in the well decreases as the head decreases it takes nearly twice this time, or about 12 seconds, to reach equilibrium. For large tides, such as at Liverpool, the rate of rise is never greater than 0·002 foot per second, so that with the relative dimensions cited there will never be any appreciable lag in the well as against the tide. For tides of small range the orifice could of course be reduced, but a small orifice is more likely to get choked than is a larger one.

We can use the above formula to illustrate the effect of the orifice in reducing surface waves. The periods of surface waves vary considerably, but we may take 4 seconds as about the normal. In half this period the wave will have passed from mean level to maximum elevation and back to mean level. Hence if we ignore the diminution of the wave effect with depth below the surface, the formula given above indicates that if the wave rises one foot above water level in the well in one second and falls again in the next second, then the rate of rise of the water in the well (with the dimensions quoted) will vary from 0·05 foot per second when the wave is at maximum elevation to less than 0·00 foot per second when the wave has returned to mean level and water level in the well is the higher. That is, within the half

period the rise in the water surface in the well will be about 0·04 foot. If we also allow for the decrease in movement with depth below the surface it is clear that even large waves will have very little effect in the well.

10.5. Dimensions of float

The accuracy of a record of the heights of the tide depends in the first instance on the float and counterpoise. The power available is of course provided by the tide; when the tide is falling the float falls, and in falling actuates the mechanism and raises the counterpoise, thus storing power; when the tide is rising the float rises and the mechanism is actuated by the falling counterpoise. When there is no tidal movement and the system is in equilibrium, the counterpoise pulls the float upwards and the water line on the float is rather lower than it would be if the float were free; the float consequently exercises a downward pull on the wire which exactly balances the upward pull of the counterpoise, and which is equal to the difference between the weights of water displaced by the float in its free and equilibrium positions, or to the weight of water displaced by the section of the float between its free and equilibrium water lines. If the tide now begins to fall, the water line on the float falls till the downward pull of the float, which is increased by the further decrease in its displacement, becomes sufficient to overcome the upward pull of the counterpoise and the frictional forces of the mechanism; the apparatus then commences to register the falling tide. Similarly if the tide begins to rise from the equilibrium position, the mechanism does not register till the water line on the float has risen so far that its decreased downward pull allows the counterpoise to overcome the remaining downward pull and the frictional forces of the mechanism; the decrease in the downward pull of the float is equal to the weight of water displaced by the section of the float between its equilibrium water line and its water line when the mechanism begins to register the rising tide. The mechanism thus lags behind the tide both when the water is falling and when it is rising; the error in the height registered is in both cases equal (on the scale adopted) to the difference between the water line on the float in the equilibrium position and the corresponding water line when the tide is falling, or when it is rising.

If the diameter of a cylindrical float is d inches, and the difference between its water line in the equilibrium position and its water line when the tide is falling, or rising, is h inches, then the volume of water displaced by the section of the float between the water lines is

$$0{\cdot}25\,\pi d^2h = 0{\cdot}8\,d^2h \text{ cubic inches} \qquad (10.5a)$$

A cubic foot of sea water weighs about 64 lb., and a cubic inch $64 \times 16/12^3$ ounces so the weight of this volume of water is

$$0{\cdot}8\,d^2h \times 64 \times 16/12^3 = 0{\cdot}5\,d^2h \text{ ounces} \qquad (10.5b)$$

Suppose we find that a weight of 8 ozs. suspended from the circumference of the gauge wheel is required to operate the recording mechanism and that the maximum error of the record must not exceed 0·2 in., then

$$0{\cdot}5\,d^2 \times 0{\cdot}2 = 8$$
$$d = 9 \text{ inches, approximately}$$

An approximate formula giving the error of the gauge in inches is readily deduced to be

$$2{\cdot}15\,W/d^2 \qquad (10.5c)$$

where W = weight in ounces required to actuate the mechanism
d = diameter of float in inches

and from this formula we obtain the following table showing the error of a gauge in inches according to the diameter of the float and the weight required to operate the mechanism.

TABLE 10.1

Error of Gauge

(in inches)

Diameter of float in inches	Weight required to operate mechanism (in ounces)					
	4	8	12	16	20	24
4	0·54	1·08	1·61	2·15	2·69	3·23
8	0·13	0·27	0·40	0·54	0·67	0·81
12	0·06	0·12	0·18	0·24	0·30	0·36
16	0·03	0·07	0·10	0·13	0·17	0·20
20	0·02	0·04	0·06	0·09	0·11	0·13

The table shows that large diameter on the water line is of primary importance for the reduction of error in the record of the tide. There is no advantage whatever in having a long cylindrical float, and experience shows that the most suitable type is a short copper cylinder, 4 to 6 inches high, with the top rounded for strength, in fact " bun shaped." A large copper ring for the suspension is useful in case the wire breaks, for it is then necessary to fish for the float from the top of the well. The diameter of the float should not be less than about 8 inches for the smallest gauge. The float should be weighted so that about half of its vertical sides are immersed when in use. Many gauges are now fitted with 20 inch, or even 24 inch, floats; a 20 inch float with 6 inch sides weighted so that half its sides are immersed weighs about 36 pounds.

10.6. The height recording mechanism

A mechanism for recording the height of the tide (see Fig. 10.1) is usually operated from a float by means of a wire or metal band round the gauge wheel. A second wire or band passes round a small wheel keyed to the shaft of the gauge wheel, and this actuates the carriage of the pen which records the height of the tide.

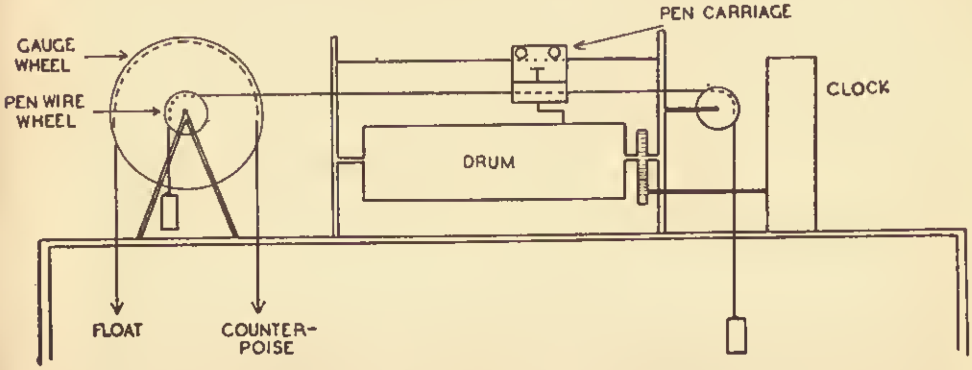

FIG. 10.1. Diagram of automatic tide gauge.

It is necessary to keep these wires taut by suitable weights or counterpoises, and, in the case of the float wire, special consideration needs to be given to the arrangement of the counterpoise system. Two methods are in common use. In one method (Fig. 10.2), the counterpoise is arranged to rise and fall in a small well which is separated from the main well in which the float rises and falls. The counterpoise is then necessarily below water level near high water and if it is immersed in water the alteration in its effective weight affects the recording mechanism. The

counterpoise well must therefore be watertight ; if the gauge is situated on a structure in the sea, a length of iron drain pipe plugged at one end makes an effective well for the counterpoise.

The other method of arranging the counterpoise (Fig. 10.3) is by the use of a system of pulleys which render unnecessary the use of a special well for the counterpoise. In this method it is necessary to take account of the friction in the pulleys.

The counterpoise should be sufficiently heavy to actuate the mechanism when the tide is rising, but not so heavy as to lift the float to an excessive degree in the water when the tide is falling. No fixed rule can be given regarding the weight,

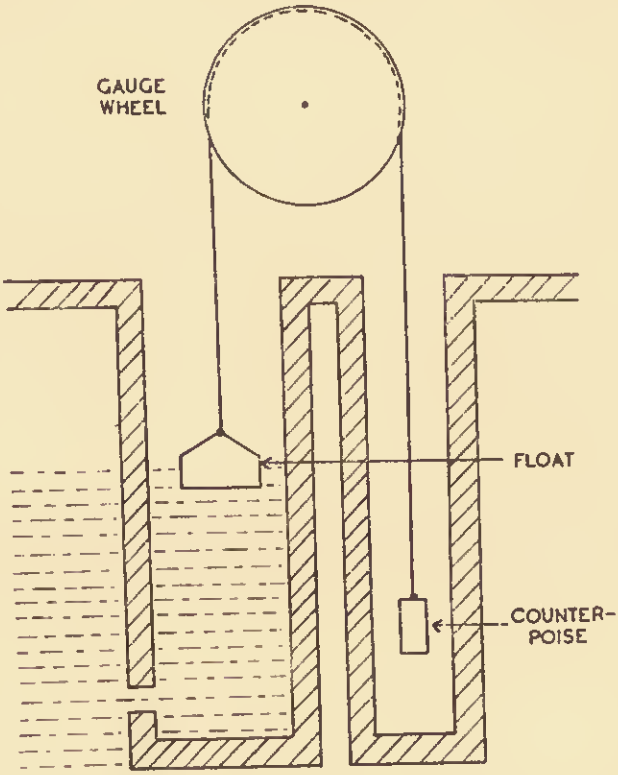

FIG. 10.2. Design of counterpoise : wire and separate well.

but it is better to have a counterpoise which is a little heavier than is actually necessary rather than one which is too light.

If a wire is used, then a complete turn must be taken round the gauge wheel in order to prevent slipping. An alternative method is to use a metal band, with perforations which are engaged by sprockets on the circumference of the gauge wheel. This method evades all risk of slipping, but complications arise owing to the need for providing for the variations in the weight of the band on the float side of the wheel. Fig. 10.4 shows a usual method of compensation ; the band, as it comes off the gauge wheel, is coiled round a gathering wheel by the action of the counterpoise. The counterpoise wire runs in grooves in a cone keyed to the shaft of the gathering wheel ; the wire is at the base of the cone at low water and at its apex at high water, so the counterpoise acts with greatest effect when the maximum length of band is out.

Similar considerations apply, in less degree, to the pen wire. A complete turn of the wire should be taken round the pen wire wheel, to prevent slipping. A light

copper or German silver band may be used instead of the wire, with perforations which are engaged by the sprockets in the circumference of the wheel.

The use of metal bands is to be commended, but gauges fitted with bands are necessarily more expensive than those which use wires.

The pen must be kept in contact with the paper either by gravity or by a light spring, and the pen carriage should be capable of adjustment relative to the wire or band which actuates it in order to set the gauge accurately.

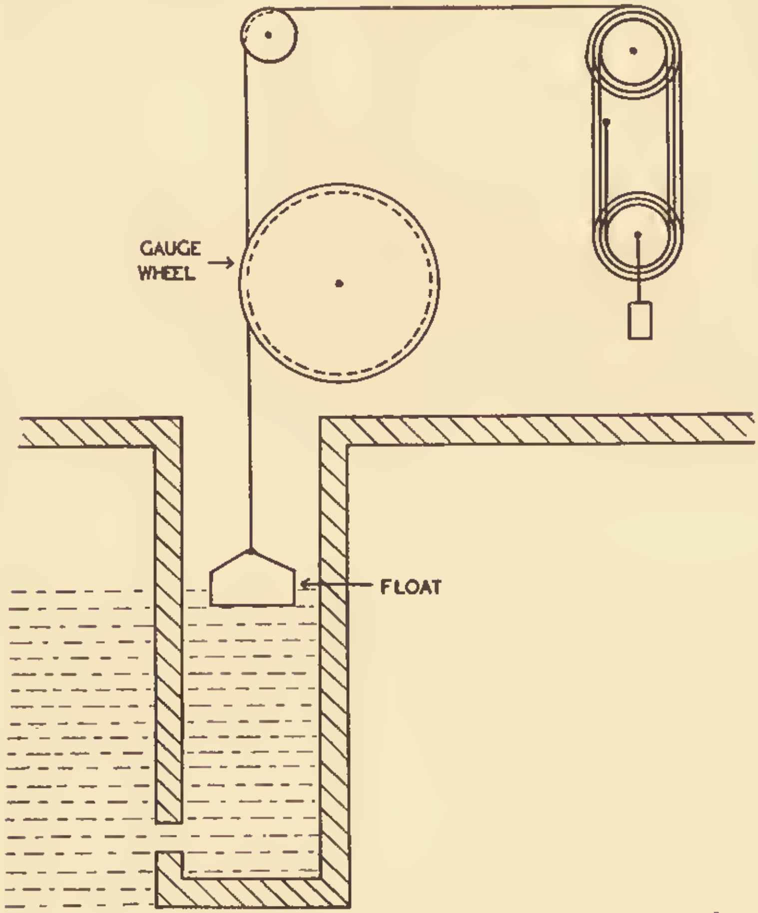

FIG. 10.3. Design of counterpoise: wire and pulleys.

Frictional effects can be detected by direct observation of the height of the water in the well at the same level on the rising and the falling tides, and comparing these heights with those registered by the gauge. If the gauge is correctly set the errors will be equal and opposite; the gauge should always be so set after its errors have been found by observation. If the frictional forces are great, or if the float and counterpoise are unsuitable, the visible effect in the tide gauge diagram is the appearance of "flats" near high and low water. The gauge always lags behind the tide, so when high water is reached the gauge indicates a height which is too low by the amount of the lag; it then ceases to register changes in the height till the tide has fallen an amount equal to the lag below the height registered by the gauge. In the meantime the drum (see Art. 10.7) continues to rotate and the pen draws a straight horizontal line on the diagram. Flats occur at low water in

the same manner. It is sometimes supposed that flats show the stand of the tide; this is not true for the tide commences to fall immediately that it ceases to rise, and commences to rise immediately when the fall has ceased.

If the frictional forces are very great and the power developed from a small change in the height of the tide is insufficient to overcome them, " steps " will occur in the tide gauge diagram both when the tide is falling and when it is rising. During the falling tide, for instance, if the mechanism is not actuated till the tide has fallen

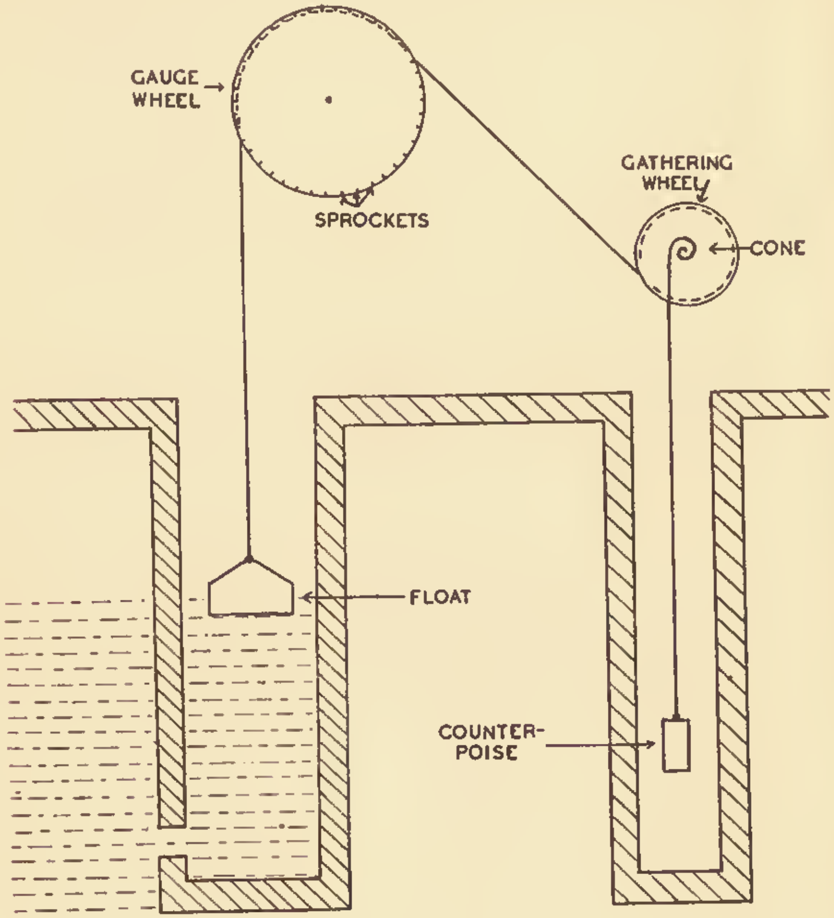

Fig. 10.4. Design of counterpoise: metal band, sprockets and compensating cone.

so far that the float is nearly clear of the water, the float will gain momentum when it does fall and will continue to fall till it is submerged to a line above the equilibrium water line, after which it will rise, leaving the wire slack and no further fall will occur till it is again nearly clear of the water. Such steps may continue during the whole period of the fall and may also occur when the tide is rising.

Tide gauge diagrams in which flats appear are unreliable; those with steps are valueless. There are but few working parts in a well-designed automatic tide gauge and, provided the gauge is not defective, flats and steps can usually be traced to the use of a float which is too light and of too small diameter and a counterpoise which is too light. They can usually be cured by fitting a heavier float of large diameter and a heavier counterpoise.

Direct readings of the height of the water in the well of an automatic tide gauge should be obtained daily, by the electrical device or other method ; these readings should be entered on the diagram when it is removed from the gauge. If the observation is made at about the same time each day a continuous record of errors at all states of the tide will become available for the use of the computer when the harmonic constants, etc., are evaluated.

10.7. The time recording mechanism

A mechanism for recording the time consists of a clock geared to a drum which it rotates under the pen (actuated by the rise and fall of the tide). The drum should be rotated once in 24 hours exactly ; if, as in some gauges, the period of rotation is seven days the time scale is too small.

The clock should run for eight days without re-winding and must be of sufficient power to rotate the drum and keep correct time throughout the period. It should be compared with a standard clock daily and the errors, if any, noted on the gauge diagram when it is removed from the gauge. Correct time keeping is of great importance ; if the range of the tide is 12 ft., then at about half tide movement is about 0·05 foot per minute, so that a time error of only two minutes results in an error of 0·1 foot in height at the true hour. Such errors can be corrected when reading the gauge diagram but correction is laborious.

The paper on which the tide curve is drawn may be either a continuous roll or a single sheet. In the former case, the paper passes from one container to another over the drum, the driving force of the clock being transmitted to the paper by needle points at each end of the drum. The perforations made by the needles serve to indicate the hours, a double perforation being made at zero hour each day. If a single sheet is used, it must be wrapped round the drum and secured by clips, and the free edges pasted or gummed together. It is an advantage if the sheets are supplied with a gummed edge.

The continuous roll is always a roll of plain paper, usually long enough for four weeks' record, and there are many disadvantages associated with it. The reading of the record is laborious, as all the time lines must be ruled by hand, by joining the pricks at the top and bottom of the paper, and the height where the tide curve crosses each time line must be obtained by measurement. Also the perforations are not always easily visible, and the rolls are awkward to handle. A second pen, for drawing the zero line of the record, is required on the gauge. The tidal record on the parts of the roll which have passed over the drum is not available till the whole roll has been used. A really strong and efficient clock is necessary to perform all the operations required of it.

Single sheets have many advantages. The sheets are usually graduated for time and height and so are easily read ; the line drawn by the pen is always visible, and past readings may be obtained if required. The tides are progressively later, on the average, by about 50 minutes each day so that a record for a week, or even two weeks, can be taken on a single sheet. Changing the sheets presents some difficulty, particularly with a large sheet ; the used sheets must be removed from the drum by loosening the clips and slitting the pasted edges with a sharp knife. The new sheet is then wrapped round the drum, the clips tightened and the free edges pasted down. The greatest care is necessary to see that the height lines agree exactly where the edges are joined. With large sheets it helps to slip two or three elastic bands of suitable size round the drum after wrapping the sheet round it ; the edges of the paper can then be adjusted before tightening the clips ; the bands should be removed when the clips have been tightened and the edges pasted down. The drum must always be removed from the machine before changing the sheet ; many makers supply two drums with each gauge and a special holder in which to place the drum whilst the sheet is being fitted to it.

After the sheet has been changed, the drum must be set to agree with the time shown on the clock, and the pen to agree with the height of the water in the well.

10.8. Electrical recording gauges

A mechanically operated gauge must of necessity function very near to the tide well, but it is often inconvenient to have offices near the gauge, so that if frequent reference to the gauge is necessary, as at Dock and Harbour offices, electrical registration is sometimes used. An electrical indicator is quite easily operated by a gauge. It is only necessary to have a drum mechanically geared to the wheel which is operated by the float and wire system ; this drum in its simplest (but least efficient) form can have on its periphery a closely wound resistance wire so that a contactor to the drum serves to record at the distant offices the voltage between the contact and the zero, this voltage varying with the position of the contactor, and therefore with the tide. Such a system is liable to error due to variations in the voltage so that some kind of test system is necessary at the office in order to regulate the voltage. There are many possible variations of this type of indicator.

There are many other types of electrical recorders. At first sight it would appear to be a very simple matter to cause the float and counterpoise system to operate a switch which would send an impulse to an indicator or recorder for every inch rise of the water, but experience teaches that the amount of power required to operate a switch is surprisingly great. It must be rendered impossible for the switch to remain partly closed such as might happen at high water, and extra power is required for this provision, while again more power is required for any throw-over device after high or low water. As a rule it can be taken that any direct method of this sort is beyond the power of the float within the limitations of error permissible.

One method of overcoming the difficulty is to spread the load over the whole period of rise, and a revolving arm is lifted by gearing until it tumbles over freely, and in falling operates a switch. The throw-over can be operated by means of a differential gear. There are many ingenious mechanisms which have been designed for the purpose of transmitting signals, but the principles involved have been sufficiently indicated above.

The transmitted signals may pass through electro-magnets and the armatures of these magnets operate ratchet gears so that at each signal a toothed wheel is knocked round by one tooth. Then through suitable gearing the total effect may be indicated on a dial. The last wheel of the system can be used to operate a pen carriage on a drum in the same manner as that described in the preceding articles. Where the electrical signals are transmitted only over short distances the power available may be sufficient to work the electro-magnets, but otherwise the signals must operate a sensitive relay, and local electrical currents be thereby used to operate the electro-magnets.

It is evident that there are many sources of error ; the switches may not operate owing to dirt, or if the switches break a circuit instead of making one, then dirty contacts may be responsible for spurious signals. The relay may fail, the electro-magnets may be imperfect. It is necessary therefore to make provision for direct and independent tests of the actual elevation. This is best effected by a separate indicator of the type referred to above.

It is possible to have automatic correction so that an indicator at the recording end is maintained in step with a drum at the transmitting end, but the system is necessarily very complicated, and requires several wires so that it cannot be operated over long distances.

Alternating current can also be used, so that as a float rises it alters the inductance in an apparatus hanging over the float. This apparatus is then withdrawn electrically until equilibrium is again reached. Such systems can only be operated over short distances.

It is just as necessary with electrical gauges as with mechanical ones that tests be made by direct sounding in the well.

The advantages of electrical systems are almost entirely in connection with the convenience of having records in offices at some distance from the tide well. The disadvantages are very great in other respects. The record must of necessity be intermittent, and it must frequently be tested for faults. For accurate work, par-

ticularly in connection with land surveys requiring accurate mean sea level determinations, electrical gauges cannot be commended.

10.9. Simple pressure gauges

As the tide is a " long wave " (see Chapter XVII) the pressure at any fixed point below the surface of the sea is proportional to the height of the water surface above the point. The changes in the pressure as the surface rises and falls consequently provide a means of measuring the rise and fall of the tide.

As a general rule pressure gauges are not considered as serious rivals to the visual and float gauges previously discussed. It is only when the coastal formation is such that a visual gauge cannot be erected and circumstances do not permit of the establishment of a recording float gauge that pressure gauges might be used. In the survey of an open coast, for instance, consisting of steep high cliffs against which the sea breaks, or even of a very wide shelving beach, the erection of a visual tide gauge is difficult or impossible. In similar circumstances also the great cost might prohibit the excavation of a well on the land, or the erection of a special structure in the sea on which a float gauge could be established, but a recording pressure gauge could be erected at small cost.

Much ingenuity has been expended in designing gauges which depend upon changes in the pressure at a fixed point below the surface of the sea. In some ways the simplest possible type of pressure gauge consists of a long flexible pipe of fairly large diameter one end of which is fixed to an anchor or sinker on the sea bottom and the other end taken to an air pump on board a ship or on shore. If air is pumped into the pipe the pressure will rise till, at a pressure depending on the depth to which the sea end of the pipe is submerged, air escapes and the pressure ceases to rise. The pressure at which the air escapes can be measured by an ordinary Bourdon gauge and the tidal movement computed from a series of such observations. Alternatively the pressure gauge may be replaced by a mercury manometer, a U tube into which mercury has been poured. One end of the U tube is connected by a T piece with the air pump and with the flexible pipe, the other end of the U is open to the atmosphere. The difference in the level of the mercury in the two arms of the U is a measure of the pressure in the system. There are other variations in the use of mercury manometers.

Another method is to connect a pressure gauge or mercury manometer, on board a ship or on shore, with an inflated rubber bag, contained in a strong perforated iron case, on the sea bottom, by a flexible pipe of very small diameter. The bag must be so large, and the diameter of the pipe so small, that the air contained in the pipe is very much less than that contained in the bag, for only the air in the bag is exposed to the changes in the pressure. As the water rises and falls, the pressure gauge will indicate changes in pressure on the bag on a reduced scale, depending on the relation between the air in the bag and the total air contents.

Yet another type of gauge depends upon the changes in the volume of trapped air in a vessel open to the sea. It is not, however, a simple matter to measure changes in volume in a direct manner.

The pressure of the sea water varies with changes in the specific gravity of the water, though for practical purposes this variation may often be neglected, but the readings of all the devices described require to be corrected for changes in atmospheric pressure when a pressure gauge is used, for then only the sea end is exposed to such changes ; correction is not required if the manometer is used, for the mercury also is exposed to these changes. The devices must also be calibrated beforehand against a visual tide gauge, and it is important that all circumstances, such as length of pipe, depth of water, etc., should be as when the device is erected for use.

It must be admitted that the simple types of apparatus here described have not been very successful in practice. The manometer types of apparatus are inconvenient in use, and the types dependent on changes of trapped air fail owing to the condensation of water in the very narrow pipes necessarily employed. Other types of apparatus have their own peculiar troubles and their use is much more troublesome than would appear on first consideration.

10.10. Recording pressure gauges

If it is desired to obtain automatic records by means of a pressure gauge, it is necessary to use very carefully designed apparatus. Recording pressure gauges are available for lowering to the bottom, and probably the most successful of these is the Marégraph Plongeur, designed by M. Favé of the French Hydrographic Service. This instrument uses two Bourdon tubes as the elements subject to pressure, and to the free ends of these tubes are attached sharp needle points which rest lightly on a circular smoked glass plate, which is rotated by clockwork. As the plate rotates each needle traces a curve which records the changes in pressure. Two tubes are used so as to eliminate certain sources of error ; they work in opposite directions and the differences in their readings, as measured on the curves, are taken. On completion of the record the glass plate is removed from the gauge and read by means of a powerful microscope. The gauge is small and compact and is enclosed in a cylindrical air and watertight case, with the Bourdon tubes only exposed to pressure from the outside.

In all automatic recorders of this nature there is a certain amount of trapped air, whether it is outside the pressure element as in the marégraph, or is part of the pressure element as in other types of gauges. As this volume of air is very sensitive to changes in the temperature it is necessary to record variations of the temperature and also to make appropriate corrections to the indications yielded by the gauge. There are many ways of recording temperatures; photographic recording is very usual, but in the marégraph the record is made by a bi-metallic strip. If strips of two metals which expand at different rates as the temperature rises are riveted together they can only adapt themselves to changes in temperature by curling in the arc of a circle, the metal which has the greater expansion being on the outside, that is on the arc of a circle of greater diameter than that of the inside strip. In the marégraph the bi-metallic strip is fixed at one end and the movement of the free end is recorded on the glass plate by a needle point, as in the case of the pressure element.

The permanent pressure on the sea bottom in deep water is so great that a Bourdon tube constructed to indicate it would not be sufficiently sensitive to record the proportionately very small changes in the pressure due to the tidal movement. In the marégraph by an ingenious device air is forced into the apparatus outside the tubes as the gauge is lowered in the water ; consequently the pressure in the instrument outside the tubes remains the same as the pressure to which the tubes are exposed, and no change of pressure is recorded. When the marégraph is to be used in deep water it is placed with a large rubber air bag in a perforated iron case ; the air bag and the gauge are connected by a rubber tube which passes through a valve kept open by a weight suspended from it. As the case is lowered the increasing pressure of the water forces air from the bag into the gauge outside the tubes so that pressure inside the gauge remains equal to the pressure of the water outside. When the weight touches the bottom the valve closes automatically and the Bourdon tubes commence to record changes in the pressure. On raising the gauge the reverse cycle of operations is gone through.

The De Vries-Smitt gauge of the Netherlands Hydrographic Office utilises the same general principles as the marégraph, but is more compact and simpler in operation ; further, certain defects inherent in the marégraph, due to friction and to the great pressure under which the recording mechanism may be required to work appear to have been overcome.

Another type of pressure gauge for use at sea depends upon the changes of volume of trapped air, the amount of air initially in the recording apparatus being adjusted to give the required range of movement of the recording element according to the depth at the place where it is to be used. Photographic records of the movement of a bubble are used in another type of recorder.

It is necessary to correct the readings of all gauges of these types for variations in the atmospheric pressure, but this can be effected by using readings obtained on the surface or from synoptic meteorological charts.

When in use these gauges must be adequately buoyed so as to ensure their recovery ; if used in frequented waters the buoy rope may be cut by the propeller

of a passing vessel and the gauge lost. If the gauge is not upright the record may be affected; if used on a sandy bottom where there is a current it may become buried in sand and cease to register. Owing to their delicacy, gauges of these types are more suitable for use by trained scientists than in surveying ships, though given favourable conditions good results may be obtained by those without much preliminary training.

10.11. Current observations

To obtain accurate measurements of the current is not a simple matter, for the observations are necessarily obtained from a floating station, which is itself subject to movement. Probably the earliest method of observing was to measure on the deck of a ship at anchor the drift in a specified time of a chip of wood or other small floating object thrown overboard from the forecastle, and to judge the direction of its drift; observation by log and log line is a development of this method. Subsurface currents to a depth of about 30 to 35 ft. may be observed in a similar manner if the log ship is replaced by a large submerged kite suspended from a small indicator buoy on the surface; a light spar 30 to 35 ft. long, weighted at one end so that it floats vertically with the other end just showing above the surface, used in place of the log ship gives the average current in the upper stratum of water, by which surface navigation is affected; observations by the latter method are specially valuable.

Observations of the drift of an object from a ship at anchor give the drift relative to the observation spot on board the ship; that is, they give the resultant of the movements of the observation spot and the object; such observations should therefore be obtained only when the ship is lying head on to the current. With strong winds blowing across or against the current the movements of the observation spot may be so great that the observations are valueless.

If near the land in well surveyed waters, where changes in position can be determined exactly, the surface current can be determined by drifting in a boat and fixing the position from time to time; a small floating object, unaffected by the wind, followed in a boat and fixed from time to time gives better results.

In deep water a small buoy can be moored by sinker and sounding wire and observations of the surface current obtained by measuring the drift from the buoy of a small floating object; the subsurface current to a depth of 30 to 35 ft., and the average current to that depth, may also be observed by submerged kite or spar, as described above.

10.12. Current meters

Currents may also be observed by means of instruments specially designed for the purpose; these are so numerous that only general principles can be expounded. Some meters record the direction and velocity of the current continuously; others must be lowered to the specified depth for each observation and brought back on board the ship for the results to be read off. All meters when in the water are kept in the direction of the current by a long fan or tail piece.

The measurement of velocity is relatively simple as compared with the measurement of direction and it is effected in all types of meters by one of two methods; either the use of a continuously revolving propeller whose revolutions in a specified time are counted automatically, or by the deflection of a pressure plate. The counting of the revolutions can be made by a counter, like the counter on a bicycle wheel, from any given moment after the apparatus is lowered, by dropping a "messenger" down the suspension cable; this engages the gears, and the counting can be stopped by a similar device. Some instruments give indications of revolutions electrically by causing a clicking noise in a telephone; the telephone wire is of course part of the cable suspension. There are very many methods of counting revolutions and also of recording them on a revolving drum.

Pressure-plate recorders use the principle that the pressure varies as the square of the velocity of the current. Apparatus of the Bourdon tube type can be used, or the movement of a plate hanging vertically by gravity can be magnified by a suitable

gearing and recorded on a revolving drum. Alternatively the pressure plate may be like a propeller in appearance, but the revolution is hindered by springs. In this case the pressure is used to rotate an arm which gives electrical indications of its position relative to a potentiometer ring. As the arm changes, the readings of a galvanometer on board the ship can be read off or recorded.

The indications of direction in most meters now in use depend on the magnetic compass: the gyro compass appears not to have been utilised. The movements of the compass needle can be recorded on a drum, or the needle may be freed by the messenger which starts the counting mechanism of the propeller, and fixed again by that which stops the mechanism. It is important to note that the power available in a magnet needle is extremely small. Many ingenious instruments designed to indicate both direction and velocity use small steel balls which drop from a reservoir at intervals corresponding to a specified number of revolutions of the propeller, and they are directed along a chute attached to the compass needle so that they are deposited in one or other of a number of compartments. The number of balls resting in the various compartments can be used to indicate the principal direction of the current as well as its magnitude.

In all cases the use of a current meter involves many difficulties. The meter must be suspended from a ship at anchor, though a recording meter can be suspended from a buoy, or from a span between two buoys. The depth of water in which the meter can be used is consequently limited to that in which anchoring is possible, and the meter is likely to foul the moorings when the current changes direction and to suffer damage. The apparatus for recording velocity must be calibrated before the meter is used; this should be done by the maker and the necessary data supplied with the instrument. The compass indications will be affected by the magnetism of the ship observing unless the depth at which the meter is used is so great that it is outside the ship's field of force; tests of errors of direction so arising show that they may be very great if the meter is close to the ship.

The current meter records the resultant of the passage of the water past it and its own movements in the water, due to the movements of the ship from which it is suspended. The ship movements may be very great, particularly when the relative wind and current are such that the ship is lying across the current. The meter movements are less than the ship movements, and decrease with the distance of the meter below the ship; observations by meter are consequently usually considered to be more precise than those obtained from the ship by the methods first described. It must, however, be remembered that when observations are obtained from the ship the observer knows whether they are reliable, whereas the meter indications may be incorrect owing to maladjustment of the instrument or other unknown cause. The results of movements of the meter in the water probably cancel out in the record of a meter recording continuously, but affect each observation of a meter of the single observation type.

Note. In Arts. 10.11 and 10.12 "current" is used for the horizontal movement of the water, that is, for the resultant of the tidal stream and the current.

CHAPTER XI

NON-HARMONIC TERMS

11.1. Use of non-harmonic terms

We do not know by what method, if any, seamen of the middle ages computed or estimated the time of high water, though the connection between the moon and the tide was known from very early times. Elizabethan seamen predicted by the moon's bearing—" full sea when the moon bears south-west "—and, substituting the time interval between the moon's transit and high water for the moon's bearing, the same method has remained in use till quite recent times. The diurnal tide cannot, of course, be predicted in this manner; this tide was, in fact, formerly considered to be an irregularity due to wind and a note to the effect that " the tide is irregular and mainly due to the wind " may be found on old charts of localities where the diurnal constituents predominate. In the harmonic method of predicting, the tide is regarded as the aggregate of a number of harmonic constituents, but in the non-harmonic method, the tide is regarded as a whole, and the more important phenomena have been described by terms which it is convenient to retain even though the non-harmonic methods are not used.

Further, at the present time, there is a large amount of tidal data expressed in non-harmonic form, and until other observations can be obtained, it is necessary to show how approximate harmonic constants can be obtained from the non-harmonic constants.

For the following articles it will be presumed that we are dealing mainly with tides of the semidiurnal type in which M_2 and S_2 are the most important constituents, and with mixed tides in which the diurnal tide is not predominant. In certain places, it happens that N_2 is actually more important than S_2, but we shall ignore these abnormalities.

11.2. Semidiurnal non-harmonic terms

The most important non-harmonic relations are the time intervals between the moon's transit and the next high water and low water.

The High Water Lunitidal Interval (H.W.I.) is the time interval between the moon's transit and the next high water. The interval may be taken from the local time of moon's transit to the local time of high water, or it may be taken from the Greenwich time of the moon's transit to the standard time of high water, so that it is necessary to state precisely, or to ascertain, which practice has been followed. In case of doubt, it may be taken that until recent years the former practice was followed.

The Mean High Water Interval (M.H.W.I.) is the mean value of all the high water intervals throughout at least a lunation of 29 days.

High Water Full and Change (H.W.F. & C.) is the name given to the high water lunitidal interval on the days of full and new moon, when the moon's upper or lower transit occurs at midnight. Since a transit does not often occur exactly at midnight, the precise value of this constant was obtained by interpolation between two values before and after midnight. This quantity, though freely used, was never so convenient as the mean high water interval, but it could be approximately obtained from fewer observations than were required for the mean interval. It may be taken that this interval of time always relates to the local times of transit and high water.

Two other expressions are sometimes encountered for intervals defined as above, the *establishment* or *vulgar establishment* (= H.W.F. & C.) and the *corrected establishment* (= M.H.W.I.). These expressions are no longer officially used.

Low Water Intervals (L.W.I., M.L.W.I., L.W.F. & C.) are defined in a similar manner to the high water intervals.

Spring Tides are the greatest semidiurnal tides in a semi-lunation of 15 days. They are not necessarily the highest tides or the lowest tides, for these may be partly due to the diurnal tide. Spring tides are due to the reinforcement of the lunar semidiurnal tide by the solar semidiurnal tide, and occur when the high water of one is almost simultaneous with the high water of the other. The height of high water springs varies from month to month owing to the variations in the lunar and solar semidiurnal tides.

Mean High Water Springs (M.H.W.S.) and *Mean Low Water Springs* (M.L.W.S.) are the average values derived from a sufficiently long series of observations of high water springs and low water springs.

Neap Tides are the semidiurnal tides of smallest range in a semi-lunation of 15 days. They are not necessarily the tides of smallest range, as diurnal tides may also be operative in producing these. The neap tides occur when the solar semidiurnal tide is nearly at low water when the lunar semidiurnal tide is at high water. The range of the neap tide varies from month to month.

Mean High Water Neaps (M.H.W.N.) and *Mean Low Water Neaps* (M.L.W.N.) are the average values of the heights of high and low water at neap tides.

Mean Tide Level (M.T.L.) is the average value of all the heights of high and low water. This level is called M_0 in Part II of the Admiralty Tide Tables.

The Age of the Tide is the interval of time between new moon and full moon and springs, and it is approximately the same as the interval between quadrature and neaps. The age of the tide used to be explained as the time required for the tide, which was said to be generated in the Southern ocean, to reach other parts of the world, but this explanation is fallacious. The " age " of the tide is due to the fact that on the average over the earth the solar tide lags behind the solar forces by a greater amount than the lag of the lunar tide behind the lunar forces. The average value of the age is about 1 to 1½ days, but in some cases the age is actually negative.

(At the present time, no wholly satisfactory dynamical explanation has been tendered for this world-wide phenomenon. Apart from the effects of friction, in any closed basin, if positive ages occur in one part, they should be balanced by negative ages elsewhere in the basin. It is possible, though it has not been dynamically proved, that this phenomenon is due to dissipation of energy in the coastal fringes.)

The Phase Inequality is a term used to express the changes in range between springs and neaps, and again from neaps to springs, and also the corresponding changes in the high and low water intervals. During these changes, the interval of time between alternate high waters will sometimes exceed the average interval of 24 hours 50 minutes, and the retardation is called the *lagging of the tides* ; when the interval of time between alternate high waters is less than the average, the acceleration in time is called the *priming of the tides.* As a rough generalisation it may be said that the tide lags from neaps to springs, and primes from springs to neaps.

11.3. Diurnal non-harmonic terms

The effect of a small diurnal tide on a large semidiurnal tide is to increase the heights of alternate high water and decrease the heights of the intermediate high waters, with similar effects in the heights of low water. Similarly, alternate high waters may be accelerated in time, and the intermediate high waters may be retarded, with similar effects in the times of low water. These effects are referred to as the *Diurnal Inequality.*

Diurnal Springs and *Diurnal Neaps* are terms which have not hitherto been much in use, but obviously the diurnal tides exhibit changes in range which can be expressed by terms similar to those used for the semidiurnal tides, and no confusion need arise if the words " springs " and " neaps " are qualified by the word " diurnal " when they are applied to the diurnal tide.

When the diurnal inequality is rather large, alternate high waters may be greatly unequal in height, with similar changes in the low waters, and in the limiting

case the decrement in height of high water and the increment in height of low water, due to the diurnal inequality, may cause adjacent high waters and low waters to be nearly equal, both to one another and to mean sea level. Under these circumstances, greater attention is paid to those intermediate tides which are accentuated by the diurnal inequality, and it has been found desirable, in general, to refer to these as the *Higher High Water* (H.H.W.) and the *Lower Low Water* (L.L.W.).

The Age of the Diurnal Tide is sometimes referred to, and it is the interval of time from the maximum declination of the moon to the time of high water of the following diurnal spring tide, both times generally being local times.

11.4. Approximate harmonic constants

Any mean non-harmonic quantity has an essential simplicity in that it does not depend upon any theory as to the composition of the tide, whereas the exact expression of the same quantity in terms of harmonic constants may be somewhat difficult and complicated. For many cases, however, particularly if shallow-water constituents are ignored, the non-harmonic quantities can be very simply expressed in terms of harmonic constants. We shall proceed to state a few important relations, and then in Arts. 11.5 to 11.9 to give more exact values.

We shall use the following notation :—

$$\left.\begin{aligned} Z_0 &= \text{mean sea level} \\ M_2 &= \text{amplitude (H) of } M_2 \\ m_2 &= \text{speed of } M_2 \\ g(M_2) &= \text{phase-lag } (g) \text{ of } M_2 \\ \kappa(M_2) &= \text{phase-lag } (\kappa) \text{ of } M_2 \end{aligned}\right\} \quad . \quad . \quad . \quad (11.4a)$$

with a corresponding notation for the constituents S_2, K_1 and O_1. It will be supposed that the phase-lag is expressed in degrees of angle, and that the speeds are in degrees per mean solar hour.

If the interval is taken from local time of the moon's transit to the local time of high water

$$\text{M.H.W.I.} = \frac{\kappa(M_2)}{m_2} \text{ hours} \quad . \quad . \quad . \quad . \quad (11.4b)$$

but if the interval is taken from the Greenwich time of the moon's transit to the standard time of high water

$$\text{M.H.W.I.} = \frac{g(M_2)}{m_2} \text{ hours} \quad . \quad . \quad . \quad . \quad (11.4c)$$

Since the shallow water constituents are neglected, then mean tide level and mean sea level are approximately the same, so that

$$\left.\begin{aligned} &\text{M.H.W.S. (or spring rise)} &&= Z_0 + (M_2 + S_2) \\ &\text{M.H.W.N. (or neap rise)} &&= Z_0 + (M_2 - S_2) \\ &\text{M.H.W. (or mean rise)} &&= Z_0 + M_2 \\ &\text{M.L.W.} &&= Z_0 - M_2 \\ &\text{M.L.W.N.} &&= Z_0 - (M_2 - S_2) \\ &\text{M.L.W.S.} &&= Z_0 - (M_2 + S_2) \end{aligned}\right\} \quad . \quad . \quad (11.4d)$$

Also we have

$$\left.\begin{aligned} \text{Mean range} &= 2M_2 \\ \text{Spring range} &= 2(M_2 + S_2) \\ \text{Neap range} &= 2(M_2 - S_2) \end{aligned}\right\} \quad . \quad . \quad . \quad (11.4e)$$

The value of H.W.F. & C. depends upon the relative phase lags of M_2 and S_2, and also upon the relative values of the amplitudes of M_2 and S_2. If we write

$$\left.\begin{aligned} d &= \kappa(S_2) - \kappa(M_2) \\ D &= S_2/M_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11.4f)$$

and enter Table 3 of the Admiralty method (A.T.T., Part III) with these quantities we shall get a value of e, whence we have

$$\text{H.W.F. \& C.} = \frac{e + \kappa(M_2)}{m_2} \text{ hours} \quad . \quad . \quad . \quad . \quad (11.4g)$$

It will be noticed that in this application we have reversed the usual procedure of relating M_2 and S_2. If Admiralty Tide Tables, Part III, are not available, then a modification of the method of Art. 9.10 may be easily worked out, whence it is seen that in the above notation

$$\left.\begin{aligned} E \cos e &= 1 + D \cos d \\ E \sin e &= D \sin d \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11.4h)$$

This formula may find frequent use, as much tidal data is given in the form H.W.F. & C. Regional values of D and d, as defined above, may also be used.

$$\text{The age of the semidiurnal tide} = \frac{\kappa(S_2) - \kappa(M_2)}{s_2 - m_2} \text{ hours} \quad . \quad . \quad . \quad (11.4i)$$

$$\text{The age of the diurnal tide} = \frac{\kappa(K_1) - \kappa(O_1)}{k_1 - o_1} \text{ hours} \quad . \quad . \quad . \quad . \quad (11.4j)$$

11.5. Harmonic equivalent for mean high water interval

Let the lunar semidiurnal constituent M_2 be denoted by

$$M_2 \cos m_2 t \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.5a)$$

so that the origin of time is at the high water of M_2, and let other harmonic constituents be denoted typically by

$$A \cos nt + B \sin nt \quad . \quad . \quad . \quad . \quad . \quad (11.5b)$$

It will be shown in Art. 11.9 that high water of this compound constituent occurs when $t = T$ where

$$T = \frac{2nB}{m_2 M_2} \text{ hours} \quad . \quad . \quad . \quad . \quad . \quad (11.5c)$$

provided that M_2 is a predominant constituent ; that is, if A and B are small compared with M.

The value of B is a function of the time, because all constituents except M_4, M_6, M_8 . . . change in phase relatively to M_2 from one high water to another. Consequently the mean values of B and therefore T are zero for all constituents other than M_4, M_6, M_8. . . . This is in accordance with (11.4b) and (11.4c), which simply say that the mean high water interval is that of M_2 alone, it being noted that in the above expressions we have measured time from the time of high water of M_2.

Now let us consider the effect of M_4, by writing M_2 as

$$M_2 \cos (V_0 + m_2 t - M^\circ_2)$$

where M_2 here conveniently denotes the amplitude, V_0 is the equilibrium argument at $t = 0$, and M°_2 is the phase lag (κ or g). Then we have the constituent M_4, conveniently denoted by

$$M_4 \cos (2V_0 + 2m_2 t - M^\circ_4)$$

If now we wish to write M_2 in the form (11.5a) we must have

$$V_0 = M^\circ_2$$

whence the expression for M_4 is

$$M_4 \cos (2M^\circ_2 - M^\circ_4 + 2m_2 t)$$

and the values of A and B are given by

$$\left.\begin{aligned} A &= \quad M_4 \cos (2M^\circ_2 - M^\circ_4) \\ B &= -M_4 \sin (2M^\circ_2 - M^\circ_4) \end{aligned}\right\} \quad . \quad . \quad . \quad (11.5d)$$

Thus the values of A and B for M_4 are unchanging, and similarly for M_6, ... Hence we get, from (11.4b) and (11.5c) with $n = 2m_2, 3m_2,$ . . . and by the principle that small independent changes may be added together

$$\text{M.H.W.I.} = \frac{M^\circ_2}{28{\cdot}98} + \frac{4B_4}{M_2} + \frac{6B_6}{M_2} + \ldots \text{ (hours)} \quad . \quad (11.5e)$$

where

$$\left.\begin{aligned} B_4 &= -M_4 \sin (2M^\circ_2 - M^\circ_4) \\ B_6 &= -M_6 \sin (3M^\circ_2 - M^\circ_6) \end{aligned}\right\} \quad . \quad . \quad . \quad (11.5f)$$

11.6. Harmonic equivalent for mean low water interval

The changes in high water time due to M_4 are reversed at low water since the phase of M_2 has changed by 180° whereas that of M_4 has changed by 360°. The changes due to M_6 are the same as for high water and for low water. Hence we readily deduce that

$$\text{M.L.W.I.} = 6{\cdot}21 + \frac{M^\circ_2}{28{\cdot}98} - \frac{4B_4}{M_2} + \frac{6B_6}{M_2} - \ldots \text{ (hours)} \quad (11.6a)$$

where B_4, B_6 . . . are defined as in (11.5f).

11.7. Harmonic equivalents for heights

It is shown in Art. 11.9 that the height of high water of the constituents denoted by (11.5a) and (11.5b) is given by

$$Z_0 + M_2 + A + \frac{n^2B^2}{2m^2_2M_2} \quad . \quad . \quad . \quad . \quad (11.7a)$$

Now for any constituent other than M_4, M_6 . . . the values of A and B will vary harmonically in passing from one high water to another, and the mean value of A will therefore be zero. If we write

$$B = R \cos \theta$$

where R is the amplitude (H) of the constituent and $\cos \theta$ denotes the harmonic variation of B, then the mean value of B^2 is equal to

$$R^2 \times (\text{mean value of } \cos^2 \theta)$$

which is equal to

$$\tfrac{1}{2} R^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.7b)$$

Therefore for all constituents other than M_4, M_6 . . . the contribution to the mean high water height is given by

$$\frac{1}{4}\,\frac{n^2}{m^2_2}\,\frac{R^2}{M_2} \quad . \quad . \quad . \quad . \quad . \quad (11.7c)$$

and by the principle of superposition of small independent quantities the total effect for all constituents other than M_4, M_6, . . . is observed by taking the sum of all such terms.

From (11.5d) and similar expressions we write

$$\left.\begin{aligned} A_4 &= M_4 \cos (2M^\circ_2 - M^\circ_4) \\ A_6 &= M_6 \cos (3M^\circ_2 - M^\circ_6) \end{aligned}\right\} \quad . \quad . \quad . \quad (11.7d)$$

and with the corresponding definitions of B_4, B_6, . . . in (11.5f) we have

$$\text{M.H.W.H.} = Z_0 + M_2 + A_4 + A_6 + \ldots + \frac{2B^2_4}{M_2} + \frac{9B^2_6}{2M_2} + \ldots + \sum \frac{1}{4}\frac{n^2}{m^2_2}\frac{R^2}{M_2} \quad (11.7e)$$

where the summation sign applies to all constituents other than M_4, M_6, . . . and M_1, M_3. . . .

It will generally be sufficient to take n^2/m^2_2 to be unity for semidiurnal constituents, $\frac{1}{4}$ for diurnal constituents, and so on.

A similar formula for mean low water heights follows by reversing all contributions except A_4, A_8, . . . so that we get

$$\text{M.L.W.H.} = Z_0 - M_2 + A_4 - A_6 + \ldots - \frac{2B^2_4}{M_2} - \frac{9B^2_6}{2M_2} - \ldots - \sum \frac{1}{4} \frac{n^2}{m^2_2} \frac{R^2}{M_2} \quad (11.7f)$$

It readily follows from (11.7e) and (11.7f) that mean tide level, which is the average level of all high waters and all low waters, is related to mean sea level by the formula

$$\text{M.T.L.} = Z_0 + A_4 + A_8 + \ldots \qquad . \quad . \quad (11.7g)$$

In many cases, the values of R may not be available for all constituents, but we have two possible methods which may be used to give approximate results. The more accurate of these is to use regional values of R/K_1, R/M_2, R/M_4, R/M_6, for the constituents of each species, so that if local values of K_1, M_2, M_4, M_6 are known then the values of R may be approximately obtained for use in (11.7e) and (11.7f). Very often the value of K_1 may be inferred, but the inference of M_4 or M_6 should be done with caution as shallow-water constituents are susceptible to local variations. The second and less accurate approximation is to use the equilibrium ratios R/K_1, R/M_2, and the derived ratios R/M_4, R/M_6. This method is often very useful, leading to

$$\sum \frac{1}{4} \frac{n^2}{m^2_2} \frac{R^2}{M_2} = (0{\cdot}10K^2_1 + 0{\cdot}07M^2_2 + 1{\cdot}2\, M^2_4 + 7\, M^2_6)/M_2 \quad . \ . \ . \ . \quad (11.7h)$$

The formula given above for the effect of any constituent on high water height may be verified by using Tables 3 and 4 of the Admiralty Method. The mean values of E and L give respectively the factors to be applied to the amplitude of M_2 for a semidiurnal constituent and a diurnal constituent respectively. Thus we get results as follows :—

D	Mean Value of E	J	Mean Value of L
0·2	1·010	0·2	1·002
0·4	1·040	0·4	1·012
0·6	1·093	0·6	1·023
0·8	1·167	0·8	1·040
1·0	1·270	1·0	1·063

The factors given in (11.7e) above, corresponding to D = 1, J = 1, would be respectively 1·270 and 1·063.

11.8 Mean spring range and mean neap range

The determination of mean spring range is more complicated still. At springs we are effectively considering M_2 and S_2 as a compound constituent. We should therefore replace the amplitude of M_2 by $M_2 + S_2$, and we should also replace M_4 and MS_4 by a combined constituent using Table 3 of the Admiralty Method; M_6 and $2MS_6$ will also have to be combined. Then in the formula (11.7c) we must replace M_2, M_4, M_6, . . . by the combined constituent, and, of course, delete S_2 from the formula.

Similarly, for neaps, we ought to replace M_2 by $M_2 - S_2$ and the shallow water constituents by the appropriate compounds of M_4 and MS_4, M_6 and $2MS_6$, . . .

Generally speaking, it is simplest to compute the contributions of the shallow-water constituents by the direct use of the predicting machine, and to replace M_2 in the rest of the formula by $M_2 + S_2$, or $M_2 - S_2$.

*11.9. Mathematical investigation of harmonic equivalents of non-harmonic constants

Let the lunar semidiurnal constituent M_2 be denoted by

$$M_2 \cos m_2 t \qquad . \quad . \quad . \quad . \quad . \quad (11.9a)$$

* See par. 1, page vii.

so that the origin of time is at the high water of M_2, and let other harmonic constants be typified by

$$A \cos nt + B \sin nt \quad . \quad . \quad . \quad . \quad . \quad (11.9b)$$

Also suppose that high water of the compound tide occurs when

$$t = T \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.9c)$$

and let

$$t = T + t' \quad . \quad . \quad . \quad . \quad . \quad (11.9d)$$

When t' is very small we can write

$$\left.\begin{aligned} \cos nt' &= \cos m_2 t' \\ \text{and} \qquad \sin nt' &= \frac{n}{m_2} \sin m_2 t' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11.9e)$$

and on substituting for t in the expressions (11.9a) and (11.9b) which together represent the tide, and then using (11.9c) we get

$$P \cos m_2 t' - Q \sin m_2 t' \quad . \quad . \quad . \quad . \quad (11.9f)$$

where

$$\left.\begin{aligned} P &= M_2 \cos m_2 T + A \cos nT + B \sin nT \\ Q &= M_2 \sin m_2 T + \frac{n}{m_2} A \sin nT - \frac{n}{m_2} B \cos nT \end{aligned}\right\} \quad . \quad . \quad (11.9g)$$

Now it is obvious from (11.9f) that if high water occurs at $t' = 0$, then we must have

$$Q = 0$$

(The same result, of course, is obtained by considering the sum of the *gradients* of the harmonic terms, since high water occurs when the tidal gradient is zero.)

If also we suppose that T is small, as it will be if M_2 is predominant, then we can also write with sufficient accuracy, if m_2T is measured in radians

$$\left.\begin{aligned} \cos nT &= 1, \quad \cos m_2 T = 1 - \tfrac{1}{2} m^2{}_2 T^2 \\ \sin nT &= nT, \; \sin m_2 T = m_2 T \end{aligned}\right\} \quad . \quad . \quad . \quad (11.9h)$$

and then from $Q = 0$, we get

$$M_2 m_2 T + \frac{n^2}{m_2} AT - \frac{n}{m_2} B = 0 \quad . \quad . \quad . \quad (11.9i)$$

or

$$T = \frac{nB}{m^2{}_2 M_2 + n^2 A} \quad . \quad . \quad . \quad . \quad . \quad (11.9j)$$

Since A/M_2 is supposed to be small we can write this in the form

$$T = \frac{nB}{m^2{}_2 M_2}\left(1 - \frac{n^2}{m^2{}_2}\,\frac{A}{M_2}\right) \quad . \quad . \quad . \quad (11.9k)$$

It is generally sufficient to take the first term only, so that

$$T = \frac{nB}{m^2{}_2 M_2}, \text{ approximately} \quad . \quad . \quad . \quad (11.9l)$$

where T is in hours if m_2 and n are in radians per hour.

If we take m_2 and n in degrees per hour then we have very nearly

$$T = \frac{2nB}{m_2 M_2} \text{ hours} \quad . \quad . \quad . \quad . \quad . \quad (11.9m)$$

The value of high water height, relative to mean sea level, is equal to P, and on applying the relations (11.9h) and (11.9l) we get

$$\left.\begin{aligned} P &= M_2 (1 - \tfrac{1}{2} m^2{}_2 T^2) + A + nBT \\ &= M_2 + A + \frac{n^2 B^2}{2 m^2{}_2 M_2} \end{aligned}\right\} \quad . \quad . \quad . \quad (11.9n)$$

if we take the first approximation to the denominator.

The expressions (11.9m) and (11.9n) are those used in Arts. 11.5 to 11.7.

CHAPTER XII

DATUMS

12.1. Objects of a datum

ALL measurements obtained in the course of a survey must be referred to a zero; thus latitudes, or measurements in a north-south direction, are referred to the equator, and longitudes, or measurements in an east-west direction, are referred to the meridian of Greenwich. The zero to which vertical measurements, both of the heights on land and of the depths at sea, are referred is called the datum.

All datums should be recorded in a permanent manner by reference to fixed marks on the land, and to the best available determination of mean sea level, for comparison of such references with those obtained in earlier and later years provides information regarding changes in the relative levels of the sea and land. The careful recording of the datum in fact gives the survey a permanent value.

12.2. Mean sea level

If the tidal forces ceased to operate, the waters of the oceans and seas of the world would have a surface level known as mean sea level. We are not concerned with the shape of the surface so produced and it suffices to say that this surface would not agree with the geoid (the surface every part of which is perpendicular to the plumb line) for variations in mean barometric pressure, in mean temperature, and in prevailing winds, would continue to exist and would affect the levels. In order therefore to compute mean sea level we must eliminate the tide, and this can best be effected by averaging heights of the water observed at short intervals of time. Mean tide level, see Art. 11.2 and (11.7g), is not the same as mean sea level, for the former level is affected by the shallow-water constituents of the tide.

Daily values of mean sea level are not constant and show large irregular changes, mainly due to meteorological conditions. If monthly values for one year are computed, the differences between each monthly value and the mean for the year are much less than the daily differences and are found to be, in some degree, regular; what irregularities remain are probably also due to meteorological causes. If monthly values for a number of years are computed and compared with the mean for the whole period the effects of irregular meteorological changes will be eliminated and the differences, if analysed, will be found to have the annual and semi-annual periods of the harmonic constituents S*a* and S*sa* (see Art. 6.6); the range of these constituents will, however, at most places be considerably greater than expected and, though partly astronomical, they are probably also partly due to regular meteorological changes.

If annual values of mean sea level are compared, small differences between them will be found, and it is probable that the level is subject to fluctuations with very long periods; the analysis of values over a long period, obtained from tidal observations at Marseilles and Brest (see "Annales des Ponts et Chaussées," 1935—X), has in fact brought to light fluctuations with periods of 18⅔ years and 93 years: for various reasons the existence of these fluctuations cannot be considered as fully proved, but if they exist they may be connected with changes in the orbit of the moon (see Art. 6.7).

The difference between mean sea level and gravitational level surface is not the same everywhere and differences have been found even at places not far apart; for instance, six years' continuous tidal observations (1915–1921) obtained by the Ordnance Survey (see Art. 12.3) show that mean sea level is about 0·8 foot higher at Dunbar than at Newlyn; this agrees with other evidence which shows that there is, in Great Britain, a rise in sea level from south to north, both on the east and the west coasts.

Mean sea level is thus not, in itself, suitable for a datum and where it is used its level must be perpetuated by reference to fixed marks on the land; further, evidence regarding the upheaval or subsidence of the land derived from changes in the value of mean sea level in relation to fixed marks should not be accepted till confirmed by the continuation of the changes during a very long period of time.

12.3. Land survey datum

(See "The Second Geodetic Levelling of England and Wales," 1912–1921, H.M. Stationery Office, 1922.)

The first primary network of levelling in Great Britain was carried out by the Ordnance Survey in the years 1840–1860. The datum of this levelling was said to be mean sea level at Liverpool, but as tidal observations were obtained only for about a week during 1844 the value used was the merest approximation. These observations were obtained on a tide pole at the old entrance to the Victoria dock, Liverpool, and mean *tide* level was found to be 43·14 ft. above the datum provisionally adopted (100 ft. below the bench mark on St. John's church), but exactly 43 ft. above this datum was adopted as the datum for the levelling. This datum was found to be 4·67 ft. (another account gives 4·45 ft.) above the level of the Old Dock sill at Liverpool. The only reliable reference to the datum appears thus to be that it is 57 ft. below the bench mark on St. John's church at Liverpool.

In later years, errors were found in the 1840–1860 levelling. These varied in different parts of the country, and were greatest near Harwich, where the error was as much as about 1·8 ft. These levels are consequently not suitable for comparison with later levels in connection with problems regarding the upheaval or subsidence of the land.

A new levelling was commenced in 1912, and the datum of the new work is mean sea level at Newlyn, as computed from six years' continuous tide gauge records, 1915–1921. This datum is of course not in itself permanent but is referred to permanent bench marks in the neighbourhood of the tide gauge, and it is these bench marks which constitute the permanent record. The new levelling is of a high degree of precision and should afford a reliable basis for comparison with precise levelling carried out in future years.

The datums of the national surveys of other countries do not all depend on mean sea level; thus, datum in Ireland is the level to which low water fell in Dublin bay on 8th April, 1837 (the permanent bench mark to which this datum is referred is on Poolbeg lighthouse); datum in the Netherlands (called N.A.P.) is the level at which the water of the Zuider zee was formerly admitted to the canals at Amsterdam; datum in France (zero du nivellement Bourdaloue) appears to have been selected without reference to sea level. All these datums have no doubt been connected with mean sea level since their establishment.

In the discussion of the datum of the land survey we have not considered the many other uses of the levels obtained, in connection, for instance, with river engineering, the construction of canals, roads and railways, etc.

12.4. Datums of hydrographic surveys

The seamen of all nations prefer that the charts should show approximate minimum depths; a low water level is therefore used as chart datum except in regions, such as the Baltic, where the tide is inappreciable and the datum is mean sea level.

There is a lack of uniformity in the datums used by the nations which publish charts; in France, for instance, datum is the lowest possible low water, but on the east coast of the United States it is mean low water. Consequently, whereas in France the tide seldom or never falls to datum, on the east coast of the United States some 50% of all low waters fall to or below datum. These differences have long been regarded as undesirable and at the International Hydrographic Conference, 1926, the nations represented agreed, subject to certain qualifications by the United States, that "chart datum . . . should be a plane so low that the tide will but

seldom fall below it." It will, however, require many years for this resolution to have full effect, for datums cannot well be changed except when new charts are published or the older charts revised. Generally speaking, the datums of the Admiralty charts of the waters of the British empire, and of other parts of the world where the surveys are by British authorities, conform with the resolution quoted, but in the waters of foreign countries, where the surveys are by the national authorities, the datum of the original survey is used for the Admiralty charts.

The 1926 resolution was not the first attempt at uniformity, for the 1919 Conference suggested a formula for a datum, to be called International Low Water; this proved impracticable. There is, in fact, no simple formula which will suit all types of tides and the best possible is probably that suggested by the late Sir George Darwin for Indian waters, where it has now been in use for many years, also in other parts of the world, under the title Indian Spring Low Water. This datum is the sum of the semi-ranges of the principal lunar and solar semidiurnal tides and of the lunar and luni-solar diurnal tides below mean sea level, or, in the harmonic notation, using the amplitudes of the constituents,

$$Z_0 - (M_2 + S_2 + K_1 + O_1)$$

The formula does not always provide uniformity, for at some places, where both the semidiurnal and the diurnal tides are important, the relative times are such that the low waters coincide near springs, and other constituents may cause the tide to fall below datum, whereas at other places the coincidence occurs near neaps and the datum may be considerably below the lowest level to which the tide falls. The datum may also fail where there are large shallow water constituents. Fortunately, though it is of the greatest importance that the datum should be referred to permanent fixed marks on land and to the best available value of mean sea level, the actual level of the datum is not really of much importance provided tbat the zero of the tidal information for the place is the datum of the chart; the computed correction to the depth according to the height of the tide is then applied directly to the soundings on the chart.

The subject of datums for hydrographic surveys is covered in detail in Admiralty Tidal Handbook No. 2, Datums for Hydrographic Surveys.

CHAPTER XIII

PRINCIPLES OF HARMONIC ANALYSIS

13.1. The data for harmonic analysis

It is desirable when discussing the principles of harmonic analysis to consider the data available for analysis. In many physical problems the analyst is only concerned with relatively simple harmonic motions, in which the only oscillatory motion is made up of a primary oscillation of known period together with secondary oscillations which have periods related to the primary period in the exact ratios 1/2, 1/3, 1/4, 1/5 . . . . If the tide, for instance, were composed of only the solar constituents S_1, S_2, S_3, S_4, . . . we should have to cope with a relatively simple problem.

The principal difficulty in the analysis of actual tidal observations is due to the variety of periodic terms, the greater number of which have periods which are not numerically related to one another in a simple way. As a result of this complexity, very elaborate methods have had to be devised and reduced to simple rules for the guidance of computers.

Another important feature of the observations is that they are subject to errors of a casual nature due to meteorological causes, and these errors may be rather large at times. Methods of analysis have to take such errors into account, and in practice it is necessary to utilise many observations in order to minimise the casual errors.

13.2. Designation of a harmonic constituent

A harmonic constituent can be expressed as either a sine or cosine of the angular variable, but there are many conveniences attached to tidal usages in which harmonic constituents are expressed as cosines in the form

$$R \cos (nt - k) \qquad . \quad . \quad . \quad . \quad . \quad (13.2a)$$

where R is the " amplitude " ;
$nt - k$ is the " argument " or " the phase " ;
t is the time, generally expressed in units of a mean solar hour from some arbitrary origin ;
n is the " speed " or the increment in angle per unit of time ;
k is the non-variable part of the argument, and is called the phase-lag, or alternatively, $-k$ is the phase at the origin of time.

The angle k in the analysis is regarded as a constant angle within the period covered by the observations treated, though it may physically have a slowly varying part. The arguments of the tidal constituents described in Chapters VI and VII contain a small angle denoted by u, which varies in a period of 19 years, but for the purpose of analysing a year's observations this part of the argument is regarded as equal to its mean value in the interval covering the observations.

The angle k may also be defined as follows :—

k is the lag of the phase behind nt, the variable part of the phase.

This definition is important when considering special origins of time, for, clearly, if time is measured from an arbitrary origin then k itself will not have physical significance, which is only acquired when the phase of the tidal constituent is related to the phase of the constituent of forces giving rise to the tidal constituent.

Proper rules, of course, have to be provided to relate the phase to the standard

of reference. If the time origin is changed so that time is measured from the moment when $t = t'$, then we have

$$nt - k = n(t - t') - (k - nt')$$

so tbat the apparent lag of phase is dependent upon the time origin. When t' is chosen so as to make $n(t - t')$ equal to the phase of the standard of reference then the new lag of phase is often referred to as "*the* phase-lag."

13.3. Analysis of a simple oscillation, free from casual error

Clearly, for any given oscillation which is free from casual error, it suffices to determine R, n and k. The value of R is determinable from a curve, being equal to the maximum excursion from the mean value, or half the range between the extreme values of high and low water. If the times of two high waters are determined then the interval between them gives the period in hours, and the value of n is equal to 360° divided by this value for the period. Then k is the value of the first high water time, in hours, multiplied by the value of n, since

$$nt - k = n(t - k/n)$$

and high water occurs when $t = k/n$.

It has been supposed in the above discussion that the curve is wholly free from casual error and has been carefully drawn, but in practice graphical methods are not used very much, as they leave far too much scope for individual errors of draughtmanship. Numerical data are preferable because the work can be submitted to adequate checks. The standard form (13.2a), however, is not well suited to numerical work and it is usual to replace it by the expression.

$$\mathrm{A} \cos nt + \mathrm{B} \sin nt \quad . \quad . \quad . \quad . \quad (13.3a)$$

where

$$\left.\begin{array}{ll} \mathrm{A} = \mathrm{R} \cos k, & \mathrm{B} = \mathrm{R} \sin k \\ \mathrm{R}^2 = \mathrm{A}^2 + \mathrm{B}^2, & \tan k = \mathrm{B}/\mathrm{A} \end{array}\right\} \quad . \quad . \quad . \quad (13.3b)$$

Obviously any two values of the "curve" will theoretically suffice to give the two quantities A and B so long as n and t are known. For instance, suppose that the motion takes the values 10 and 5 for values of nt equal to 10° and 50°, then we have

$$\mathrm{A} \cos 10° + \mathrm{B} \sin 10° = 10$$
$$\mathrm{A} \cos 50° + \mathrm{B} \sin 50° = 5$$

To solve simultaneous equations of this type is a well-known type of scholastic exercise, but it is not often necessary in practice to resort to such methods. The matter is mentioned to bring out the principle that in theory two observations are sufficient, and to point out that this simple method, even with observations entirely free from errors, requires some care in choosing the observations to be utilised. If the values of phase differ by 180°, then we cannot solve the equations, since the second one is only a repetition of the first. We must therefore avoid taking observations at intervals of half a period.

The easiest method of solving the equations is to take the observations for $nt = 0°$ and 90°, for the elevations are then simply equal to A and B respectively. More generally, the most accurate numerical results will follow from choosing the two points a quarter-period apart.

13.4. Simple oscillations with casual errors

If the recorded oscillation is not perfect then the two-point methods described above are not suitable. By " errors " we describe any perturbations of the oscillation wbich is under discussion, whether these are caused by defects in instruments, human failure, or external physical causes not related to the true oscillation, such as wave action, etc. As a general principle, influences of casual errors can only be minimised by increasing the number of observations to be analysed. As an elementary principle we may apply the two-point method many times over and take the average

of the results, which means that we perform many calculations accurately and retain the errors with their full values until the last process, that of averaging, in which the errors, being supposed to be distributed positively and negatively at random, tend to cancel out one another.

Obviously, a better way is to perform the simple averaging first, if possible, and thus to minimise the amount of calculation required to determine A and B. It is one of the cardinal principles of methods of harmonic analysis to endeavour to combine the observations by elementary grouping and averaging prior to attempting to analyse.

When we have done all that is possible in grouping the observations, then we have to assume, with a degree of truth depending upon the number of the observations, that the results are entirely due to the oscillation in which we are interested. The question then arises, What has been the consequence of these processes as concerning the harmonic constituent? If we have been careless in grouping the observations we may actually have been cancelling the positive part of the oscillation by the negative part, and so get a zero result.

A simple way of treating the observations is as follows: Suppose, for simplicity, that we are dealing with 24 observations of a diurnal oscillation of period exactly equal to 24 mean solar hours. Let these observations be regarded as at hours $t = 0, 1, \ldots 23$. What would be the effect of averaging the observations in two separate groups for $t = 0$ to 11 and $t = 12$ to 23? The process of averaging would undoubtedly reduce the casual errors but what about the oscillation itself?

Since the errors are all supposed to be casual then clearly the average value of any one group of observations has no claims over the average value of any other group of observations, so far as the probable diminution of error is concerned, provided that the groups have equal numbers of observations in them. But it is a different matter when we consider the effects of the grouping on the true oscillation, which we have seen can be expressed by

$$y = A \cos nt + B \sin nt$$

If one group can be found which will make the average value of $\cos nt$ in the group equal to zero then we can determine B very simply, and similarly for A.

The values of $\cos nt$ and $\sin nt$ are given in the following table for $n = 15°$, $30°$, $45°$, $60°$, $90°$ with n, of course, equal to $15°$ per mean solar hour for the diurnal constituent, since $n = 360°/\text{period}$.

TABLE 13.1

Tables of cos *nt* and sin *nt*

	$n = 15°$		$n = 45°$	
t	$\cos nt$	$\sin nt$	$\cos nt$	$\sin nt$
0	1·000	0·000	1·000	0·000
1	0·966	0·259	0·707	0·707
2	0·866	0·500	0·000	1·000
3	0·707	0·707	− 0·707	0·707
4	0·500	0·866	− 1·000	0·000
5	0·259	0·966	− 0·707	− 0·707
6	0·000	1·000	0·000	− 1·000
7	− 0·259	0·966	0·707	− 0·707
8	− 0·500	0·866	1·000	0·000
9	− 0·707	0·707	0·707	0·707
10	− 0·866	0·500	0·000	1·000
11	− 0·966	0·259	− 0·707	0·707
12 to 23	Repeat with opposite sign			

TABLE 13.1—*continued.*

t	$n = 30°$ cos nt	sin nt	$n = 60°$ cos nt	sin nt	$n = 90°$ cos nt	sin nt
0	1·000	0·000	1·000	0·000	1·000	0·000
1	0·866	0·500	0·500	0·866	0·000	1·000
2	0·500	0·866	− 0·500	0·866	− 1·000	0·000
3	0·000	1·000	− 1·000	0·000	0·000	− 1.000
4	− 0·500	0·866	− 0·500	− 0·866	1·000	0·000
5	− 0·866	0·500	0·500	− 0·866	0·000	1·000
6	− 1·000	0·000	1·000	0·000	− 1·000	0·000
7	− 0·866	− 0·500	0·500	0·866	0·000	− 1·000
8	− 0·500	− 0·866	− 0·500	0·866	1·000	0·000
9	0·000	− 1·000	− 1·000	0·000	0·000	1·000
10	0·500	− 0·866	− 0·500	− 0·866	− 1·000	0·000
11	0·866	− 0·500	0·500	− 0·866	0·000	− 1·000
12 to 23	Repeat with same sign.					

Considering the diurnal oscillation, clearly the sum of the observations from $t = 1$ to 11 will yield a zero result when applied to cos nt (see Table 13.1), while the sum of the values of sin nt is 7·596. A similar process with the observations from 13 to 23 will yield a contribution of − 7·596 B from the oscillation. We can combine the two results, by saying that we have multiplied the values of y by + 1, − 1, 0 according to whether sin nt is positive, negative or zero, and the sum of the results, divided by 2 × 7·596 will give B.

A similar process will give A for the diurnal constituent ($n = 15°$) and exactly similar processes will give multipliers 1, − 1, 0 (Table 13.2) for the analyses of

TABLE 13.2

Multipliers for Analyses of Simple Oscillations

t	$n = 15°$ A_1	B_1	$n = 45°$ A_3	B_3	$n = 30°$ A_2	B_2	$n = 60°$ A_4	B_4	$n = 90°$ A_6	B_6
0	1	0	1	0	1	0	1	0	1	0
1	1	1	1	1	1	1	1	1	0	1
2	1	1	0	1	1	1	− 1	1	− 1	0
3	1	1	− 1	1	0	1	− 1	0	0	− 1
4	1	1	− 1	0	− 1	1	− 1	− 1	1	0
5	1	1	− 1	− 1	− 1	1	1	− 1	0	1
6	0	1	0	− 1	− 1	0	1	0	− 1	0
7	− 1	1	1	− 1	− 1	− 1	1	1	0	− 1
8	− 1	1	1	0	− 1	− 1	− 1	1	1	0
9	− 1	1	1	1	0	− 1	− 1	0	0	1
10	− 1	1	0	1	1	− 1	− 1	− 1	− 1	0
11	− 1	1	− 1	1	1	− 1	1	− 1	0	− 1
12 to 23	Repeat with opposite sign.				Repeat with same sign.					
Divisor	15·19	15·19	14·48	14·48	14·93	14·93	16·00	13·86	12	12

tidal constituents with speeds $n = 15°, 30°, 45°, 60°, 90°$ pertaining to the solar constituents S_1, S_2, S_3, S_4, S_6.

It will be noted that these multipliers yield no contributions from an arbitrary datum. This table of multipliers has many uses and it is a simple method of computation, provided that the oscillations are known to be only of the one species.

Though the formulæ are dependent upon n being an exact multiple of 15°, it may be noted that they are usable for oscillations of other periods, by altering the time intervals from mean solar hours to "special hours," that is, if the period of the oscillations is divided into 24 "hours," then the observations of these hours can be treated by the above factors and divisors.

13.5. Analysis of mixed oscillations

We have supposed that the investigation of the previous article has been restricted to the case of a simple and single oscillation perturbed only by casual errors, and we proceed to examine the problem when there are many species of constituents occurring together. Let the tidal oscillation be denoted by

$$y = A_0 + (A_1 \cos 15°t + B_1 \sin 15°t) + (A_2 \cos 30°t + B_2 \sin 30°t) + \dots \qquad (13.5a)$$

If we apply the multipliers of Table 13.2 to the values of cos nt and sin nt given in Table 13.1 we readily verify that the multipliers for $n = 15°$ and 45° yield zero result, when applied to the values of cos nt and sin nt for $n = 30°, 60°, 90°$. Similarly the multipliers for $n = 30°, 60°, 90°$ yield zero results when applied to the values of cos nt and sin nt for $n = 15°$ and 45°. The reason for this is that the functions for even multiples of 15° repeat themselves without change of sign from hour 12 whereas those for odd multiples of 15° repeat themselves with a change of sign. Hence the products in the group of hours $t = 12$ to 23 simply annul the products in the hours $t = 0$ to 11.

Similarly, because the functions for $n = 60°$ repeat themselves after $t = 5$, whereas those for $n = 30°$ and 90° repeat themselves with a change of sign, the multipliers for $n = 30°$ and 90° yield zero result when applied to the values of cos nt and sin nt for $n = 60°$, and similarly the factors for $n = 60°$ yield zero result when applied to the values of cost nt and sin nt for $n = 30°$ and 90°.

We need only investigate, therefore, the mutual effects of the species with $n = 15°$ and 45° and those for 30° and 90° and on evaluating the sums of the products we get results as shown in Table 13.3, in which X_n denotes the result of applying the multipliers for A_n to the 24 values of y, and Y_n denotes the result of applying the multipliers for B_n to the 24 values of y.

TABLE 13.3

$$
\begin{aligned}
15{\cdot}19\,A_1 &= X_1 + 4{\cdot}83\,A_3 = X_1 + 0{\cdot}333\,X_3 \\
15{\cdot}19\,B_1 &= Y_1 - 4{\cdot}83\,B_3 = Y_1 - 0{\cdot}333\,Y_3 \\
14{\cdot}48\,A_3 &= X_3 \\
14{\cdot}48\,B_3 &= Y_3 \\
14{\cdot}93\,A_2 &= X_2 + 4{\cdot}00\,A_6 = X_2 + 0{\cdot}333\,X_6 \\
14{\cdot}93\,B_2 &= Y_2 - 4{\cdot}00\,B_6 = Y_2 - 0{\cdot}333\,Y_6 \\
16{\cdot}00\,A_4 &= X_4 \\
13{\cdot}86\,B_4 &= Y_4 \\
12{\cdot}00\,A_6 &= X_6 \\
12{\cdot}00\,B_6 &= Y_6
\end{aligned}
$$

13.6. Remarks on more elaborate methods

Large volumes have been written upon formulæ for harmonic analysis, and very elaborate schedules have been drawn up for extracting, theoretically, the utmost degree of accuracy from observations. We have made it clear that where there is a real period then two quantities serve to define the constituent, and that the observations can be grouped in any convenient manner in order to diminish the casual error. If different methods yield results whose differences have any degree of importance then it is never wise to attach much significance to either reduction, and the obser-

vations should be supplemented by others. More elaborate formulæ only involve computational labour out of all proportion to any possible gain, which is rarely likely in any case to be equivalent to doubling the number of observations. We shall however give a short sketch of the principles used in these methods.

We shall take as an illustration observations denoted by y over a period of 24 hours, that is, at hours 0, 1, 2 . . . 23. These observations may be single observations for each hour or may each represent an average of many observations which can be allocated to that hour. Then as in (13.5a) we have

$$y = A_0 + (A_1 \cos 15°t + B_1 \sin 15°t) + \ldots + (A_r \cos 15°rt + B_r \sin 15°rt) + \ldots \quad . \quad . \quad . \ (13.6a)$$

where r is an integer. Multiply the observations by $\cos 15°st$, where s is an integer, and sum for the hours 0 to 23. Then the effect of this multiplication on the typical terms $(A_r \cos 15°rt + B_r \sin 15°rt)$ is to give

$$A_r \cos 15°rt \cos 15°st + B_r \sin 15°rt \cos 15°st$$

which is equal to

$$\tfrac{1}{2} A_r \cos 15°(r+s)t + \tfrac{1}{2} A_r \cos 15°(r-s)t + \tfrac{1}{2} B_r \sin 15°(r+s)t + \tfrac{1}{2} B_r \sin 15°(r-s)t$$

Thus, since r and s are integers, and if r is not equal to s, the sums of these terms for $t = 0, 1, \ldots 23$ all vanish. For example, if $r = 2$ and $s = 1$, we have to sum the values of

$$\cos 45°t, \ \cos 15°t, \ \sin 45°t, \ \sin 15°t$$

and the numerical values given in Table 13.1 show that the sum of the 24 quantities is in each case zero.

If, however, $s = r$, then all the sums vanish in the same way except that from

$$\tfrac{1}{2} A_r \cos 15° (r-s)t = \tfrac{1}{2} A_r$$

which gives

$$12 \ A_r \text{ on summation.}$$

Hence we get

$$12 \ A_r = \sum_{0}^{23} y \cos 15°rt \quad . \quad . \quad . \quad . \ (13.6b)$$

And similarly

$$12 \ B_r = \sum_{0}^{23} y \sin 15°rt \quad . \quad . \quad . \quad . \ (13.6c)$$

Many of the older methods of analysis laboriously perform the multiplications by the cosines, that is, they replace the simple multipliers of Table 13.2 by the entries in Table 13.1. The elaborate forms encountered in some of these analyses arise from the efforts made to minimise the labour of multiplication by utilising the recurrences of factors, with or without change of sign, as noted previously.

It will be noted that the use of the multipliers cos rt and sin rt gives the values A and B without the complications of having to correct X_1, Y_1, X_2, Y_2, as noted in Table 13.3, but the advantage of this is more than offset by the labour of the multiplications as against the simple additions and subtractions which are alone required in the method of the preceding article.

We can easily invent a set of simple multipliers which will avoid having to use corrections, so that one species of constituent does not affect another. If we replace 2 cos rt, 2 sin rt by multipliers ± 2, ± 1, or 0, whichever is the nearer, we obtain formulæ and divisors as in Table 13.4, which have the advantage referred to.

The theoretical multipliers of (13.6b) and (13.6c) are really based on what is called "the least square rule" and they are supposed to reduce the sum of the squares of the errors to a minimum. The fundamental idea is that large variations are associated with large multipliers, and as the approximations to the multipliers preserve this idea, then the multipliers given in Table 13.4 may be used in preference

to those of Table 13.2 by those who attach importance to the " theoretical " accuracy of tbe formulæ (13.6b) and (13.6c). The formulæ, however, cannot be used without slight modification if fifth and seventh-diurnal oscillations occur, but these hardly ever require consideration in tidal work.

TABLE 13.4

Alternative Multipliers for Harmonic Analyses of Mixed Species

t	$n = 15°$ A_1	B_1	$n = 45°$ A_3	B_3	$n = 30°$ A_2	B_2	$n = 60°$ A_4	B_4	$n = 90°$ A_6	B_6
0	2	0	2	0	2	0	2	0	1	0
1	2	1	1	1	2	1	1	2	0	1
2	2	1	0	2	1	2	−1	2	−1	0
3	1	1	−1	1	0	2	−2	0	0	−1
4	1	2	−2	0	−1	2	−1	−2	1	0
5	1	2	−1	−1	−2	1	1	−2	0	1
6	0	2	0	−2	−2	0	2	0	−1	0
7	−1	2	1	−1	−2	−1	1	2	0	−1
8	−1	2	2	0	−1	−2	−1	2	1	0
9	−1	1	1	1	0	−2	−2	0	0	1
10	−2	1	0	2	1	−2	−1	−2	−1	0
11	−2	1	−1	1	2	−1	1	−2	0	−1
12 to 23	Repeat with opposite sign.				Repeat with same sign.					
Divisor	24·520	24·520	20·484	20·484	25·856	25·856	24	27·712	12	12

13.7. The general problem of analysis of tidal constituents

In previous articles we have been considering methods of analysis applicable to a set of constituents whose speeds are exact multiples of the slowest speed, but in tidal analyses we have to consider the vastly more complicated relations of many groups of constituents. While it is true that we have diurnal, semidiurnal, and higher species existing each day, yet the exact periods of each are obviously somewhat variable, since the times of high water do not increase at a steady rate. The multipliers given in Tables 13.2 and 13.4 under such circumstances will not efficiently " isolate " (to use a technical expression) the functions A and B ; that is, instead of giving a simple value of A, for instance, for one species they will give contributions from all species, though these will be smaller than that for the main function. An example of imperfect isolation is given for A, with the multipliers of Table 13.2, since Table 13.3 makes it evident that the combination of observations denoted by X_1 includes a contribution from the third diurnal constituent, and a correction has to be made by using X_3.

While the more elaborate multipliers of Table 13.4 are, of course, a little better in this respect, yet they will inevitably fail to give proper " isolation " when the periods differ from the solar periods ; consequently, the designer of analytical methods has to decide whether the formulæ should be still further elaborated in order to separate the species to the required degree of accuracy, or whether to use corrections for one species on another.

Similar considerations apply, as we shall show, to the analysis for the constituents of any one species. It is impossible to get perfect isolation by one operation for one of the constituents, so that corrections are in any case necessary for all constituents within the species, and the procedure depends simply upon the possibility of handling simultaneously a large number of constituents.

For the very elaborate analyses of a year's observations it is definitely impracticable to cope at one time with all the constituents of all species, since the analyses, at least those carried out by the Liverpool Observatory and Tidal Institute, are made for 18 diurnal and 18 semidiurnal constituents, with, of course, other species in proportion to their importance. In such a case, it is necessary to isolate the species, and groups of observations are taken each day so as to provide pairs of functions X_1, Y_1; X_2, Y_2; . . . to which the main contributions come from the diurnal, semidiurnal and higher species respectively. (These functions, of course, are more elaborately computed than the functions X, Y, referred to in Table 13.3, though the same notation is used for convenience.) In the case of the method given in Admiralty Tide Tables, Part III, for observations covering 15 and 29 days, the number of constituents is so small that they can be handled conveniently together. The object there is to make the grouping of observations as simple as possible, so that all the analytical processes involve only simple additions and subtractions, and appropriate corrections are made at the end. Having decided this point, a little further liberty can be taken with the multipliers either on account of convenience or for ease of checking, and in the method for 15 or 29 days the multipliers are taken as 1 for X_1, Y_1, X_2, Y_2, so differing a little from those given in Table 13.2. The only reason for not having zero multipliers was that it was advantageous to use all the observations so that a check could be provided, in that the sum of the positive and negative parts, but without regard to sign, must be the sum of all the 24 quantities in each case.

13.8. The determination of daily values of mean sea level

As an example of the difficulties arising from the variable periods, and also in order to bring to notice a valuable formula, we can consider the problem of determining the mean level of the sea. It has been customary to take the mean of 24 observations ($t = 0$ to 23) at intervals of a mean solar hour, or, alternatively, to use 25 observations ($t = 0$ to 24), centered on mean moon. Neither of these methods gives complete satisfaction; the former has no contribution from any of the solar constituents, but the lunar constituents (particularly M_2 and O_1) considerably affect the results. The 25-hour method has a smaller lunar error with an appreciable solar error. The following table gives the errors by the two methods, per unit of amplitude of constituent.

TABLE 13.5

Possible Errors in Determination of Mean Sea Level

Constituent.	Amplitude of Constituent multiplied by :—	
	(a) 24 observations	(b) 25 observations
K_1 . . .	0·003	0·042
O_1 . . .	0·075	0·032
M_2 . . .	0·035	0·006
S_2 . . .	0·000	0·040
M_4 . . .	0·035	0·006
MS_4 . . .	0·018	0·024

Thus, if the amplitude of M_2 is 10·0 ft., its contribution to the error of computed mean sea level from 24 observations may be 0·35 ft., whereas from 25 observations it is 0·06 ft. Of course, on taking a month's observations, these errors are much reduced, for the contribution of M_2 will vary harmonically from day to day. The solar errors, however, will be the same from day to day, which is a strong argument in favour of using the 24-hour grouping. Neither method is ideal.

Let us consider three special groupings of y :—

(*a*) $y_0 + y_8 + y_{16}$
(*b*) $y_0 + y_5 + y_{10} + y_{15} + y_{20}$
(*c*) $y_0 + y_2$

The effect of these on any harmonic constituent of speed n is to multiply its amplitude by $\sin 12n/\sin 4n$ in case (*a*) and by $\sin 12{\cdot}5n/\sin 2{\cdot}5n$ in case (*b*) and by $2 \cos n$ in case (*c*), as is shown in any elementary text-book on trigonometry. If the observations are at intervals of a mean solar hour, then $\sin 12n$ is zero when n is an integral multiple of 15°, and the factor then vanishes provided $\sin 4n$ is not also zero, so that the factor vanishes for the constituents S_1, S_2, S_4, S_8, but not for S_3 and S_6. These results, of course, can be tested with the values of $\cos nt$ and $\sin nt$ in Table 13.1. The combination (*c*), however, gives zero result with S_6, and so we can combine the groups (*a*) and (*c*) to give

$$(y_0 + y_2) + (y_8 + y_{10}) + (y_{16} + y_{18})$$

which will give zero results for constituents S_1, S_2, S_4, S_6 and S_8, but not S_3. Since the third-diurnal constituents are always small, we need not give further special attention to them.

The combination (*b*) gives very small results with all the constituents of the lunar series, M_1, M_2, M_3 . . ., and if this can be combined with the preceding combination of (*a*) and (*c*), we shall get a very good formula, as follows :—

$$\begin{aligned} &(y_0 + y_2) + (y_8 + y_{10}) + (y_{16} + y_{18}) \\ +\ &(y_5 + y_7) + (y_{13} + y_{15}) + (y_{21} + y_{23}) \\ +\ &(y_{10} + y_{12}) + (y_{18} + y_{20}) + (y_{26} + y_{28}) \\ +\ &(y_{15} + y_{17}) + (y_{23} + y_{25}) + (y_{31} + y_{33}) \\ +\ &(y_{20} + y_{22}) + (y_{28} + y_{30}) + (y_{36} + y_{38}) \end{aligned}$$

This formula can be expressed as in Table 13.6.

TABLE 13.6

Multipliers for Mean Sea Level

Divisor = 30.

t	Multiplier	*t*	Multiplier	*t*	Multiplier
0	1	12	1	24	0
1	0	13	1	25	1
2	1	14	0	26	1
3	0	15	2	27	0
4	0	16	1	28	2
5	1	17	1	29	0
6	0	18	2	30	1
7	1	19	0	31	1
8	1	20	2	32	0
9	0	21	1	33	1
10	2	22	1	34	0
11	0	23	2	35	0
				36	1
				37	0
				38	1

and its effects on certain constituents are given in Table 13.7, showing its great efficiency :—

TABLE 13.7

The Contributions to Calculated Values of Mean Sea Level by Formula of Table 13.6

Constituent	Amplitude multiplied by
M_2	0·0006
S_2	0·000
M_4	0·003
M_6	0·002
K_1	0·000
O_1	0·002
MO_3	0·007

It will be noted that the formula extends over a *span of 38 hours, and this spreading out over a span exceeding 24 hours is a consequence of the spreading out of the periods. It is inevitable if a " daily " formula is to isolate contributions from a single species of tide (in this case the long-period species). The central value is at hour 19, but if it is desired to have the central value at hour 12, then the multipliers must be used in order for $t = -7$ to $t = 24 + 7$, *i.e.*, seven hours on either side of the day of 24 hours.

It will also be noted that some hourly observations are not used, while others are used twice. Keeping in mind that any formula must adequately perform two duties—(1) isolation of the species of tide, (2) reduction of casual errors—and since it is evident that the first of these duties is efficiently performed by the formula, it only remains to enquire whether the casual errors are properly dealt with.

The discussion of this point would involve the theory of probability, a somewhat arid field for our purposes, but there are certain points which need to be considered in relation to the special problem. The casual errors are of two kinds, apart from errors of observation (that is, due to reading the tide gauge records), which we may ignore, namely, those of very short periods, as in the case of seiche motions apparent in the record, and those of longer periods, anything from six hours to a month, due to meteorological disturbances.

The short period errors are generally reduced by using a mean curve, and since we have a divisor of 30 with the formula, in effect we adequately reduce all such errors. The disturbances covering a longer span will yield the same effects with any formula, and in such cases there is no advantage in using observations for every hour of the day.

In other words, formulæ which have substantially the same divisors will reduce the casual errors to the same degree, and with this proviso the best formula is that which makes really adequate elimination of the unwanted periodic variations.

The discussions of this article are typical of those required for all species of tides, and modern methods are not content simply to diminish casual errors by merely using large numbers of observations, but they pay great attention to the isolation of the tidal constituents ; that is, adequate corrections for periodic errors are now a prominent feature of the methods of harmonic analyses.

13.9. Daily, monthly and annual processes

The methods of analysis used some years ago involved a tremendous amount of computation. A year's observations of tides provides about 9000 observations, and we shall briefly explain how these were dealt with.

The average value of the 365 observations at any fixed hour of the day will be principally due to the solar constituents which repeat themselves at intervals of 24 solar hours. All other constituents will change in phase through multiples of 360°, and will only give small contributions in the result. Therefore we get 24 average values for hours $t = 0, 1, 2, \ldots . 23$, which can be regarded as due principally to solar constituents, as the casual errors will be negligibly small. These can be analysed by any of the methods outlined earlier in the chapter, in order to give the solar constituents, $S_1, S_2, S_3 \ldots .$

Now consider such a constituent as M_2. If the observations could be read off again at intervals of a lunar hour, then exactly the same method as above could be used, but this is impracticable. What was done was to " assign " an observation at a solar hour to the nearest lunar hour. All the observations were rewritten (itself a formidable task) according to the rules of assignment, and then the averages taken as outlined above. The series of 24 averages yielded $M_1, M_2, M_3. \ldots$ This process was repeated for every constituent and it was obviously a very laborious matter. The only attempt to reduce the systematic errors was to choose a span of observations which reduced the residual error due to M_2.

Later methods have made corrections, but the assignment method is in itself crude, and introduces unnecessary complications and errors.

* The word *span* is introduced to avoid the use of the word period in two senses, that of an interval of time and that of duration of a cycle.

Modern methods are much more economical in labour and much more efficient. The original hourly observations are dealt with by methods of grouping, analogous to those discussed in earlier articles, so as to obtain functions

X_0 for the long period constituents
X_1, Y_1 for the diurnal constituents
X_2, Y_2 for the semidiurnal constituents

and so on. The hourly observations need never be rewritten, and the operations are carried out by putting slips of paper (containing the multipliers) against the observations, multiplying and adding the products automatically on a calculating machine. The multipliers may be ± 1, 0 for the short span of observation (15 or 29 days) and ± 2, ± 1, 0 for the span of a year. In the latter case it is necessary to have the multipliers for this process spreading outside the 24 hours of the day, as in the case of the multipliers discussed in the previous article, in order to isolate the functions. When these functions are obtained we are left with about 360 values of X_2 and 360 values of Y_2, which alone need to be considered for the determination of all the semidiurnal constituents, and similarly for other species of constituents.

Now consider the function X_2, say. It contains contributions from S_2 and these will repeat themselves every day, but the constituent M_2 will give a variable contribution depending upon the phase of M_2 at the central hour of the daily span. In other words, the contribution of M_2 to the function X_2 will vary harmonically with a period of about a fortnight. Suppose, for a moment, that M_2 and S_2 are the only constituents and that we have a month's value of X_2. Let m denote the increment of phase of M_2 per 24 mean solar hours, and let T denote time in units of a mean solar day; then we can write

$$X_2 = A_s + A_m \cos m\mathrm{T} + B_m \sin m\mathrm{T}$$

where A_s is the contribution of S_2 to X_2 and the rest is due to M_2. Then in much the same way as in Art. 13.4 or Art. 13.5 we can replace $\cos m\mathrm{T}$ by ± 1 or 0, or $2 \cos m\mathrm{T}$ by ± 2, ± 1, 0 and so obtain multipliers which will give zero result when applied to A_s or to $\sin m\mathrm{T}$ and will thus yield simply a multiple of A_m. We denote these multipliers by d_2 when there are two oscillations in a month and the effect upon X_2 is denoted by X_{22}, so that in this case we have

$$X_{22} \text{ a multiple of } A_m.$$

Similarly we can obtain multipliers from $\sin m\mathrm{T}$, which we call d_b (b being the second letter of the alphabet, so that d_2 and d_b are associated multipliers) and their application to X_2 gives a function called X_{2b}, which is simply proportional to B_m.

If we want to determine A_s we only need to add the values of X_2 to give a function called X_{20}, but the isolation will not be perfect.

Now all the semidiurnal constituents can be considered in groups whose character is specified by the number of oscillations per month in the functions X_2, Y_2 as follows :—

TABLE 13.8

Group No.	Constituent	Multipliers
0	S_2, K_2, T_2, etc.	d_0
1	L_2, λ_2	d_1, d_a
2	M_2, $2SM_2$.	d_2, d_b
3	N_2, ν_2	d_3, d_c
4	μ_2, $2N_2$	d_4, d_d

and it is a simple matter to provide multipliers d_0, d_1, d_2, d_3, d_4 and d_a, d_b, d_c, d_d, which will tend to isolate these groups of constituents, so yielding functions

X_{20}, X_{21}, . . . X_{2a}, X_{2b}, . . .
Y_{20}, Y_{21}, . . . Y_{2a}, Y_{2b}, . . .

The isolation will not be perfect, but simple methods of correction can be evaluated once for all.

It will be noted that K_2 has the same group number as S_2, so that the function X_{20} contains a large contribution from K_2 as well as S_2. The reason for this is that the two constituents have nearly equal speeds, and though the phase of K_2 changes in the month it does not march through a multiple of 360° like the phases of constituents in the other groups. To separate K_2 and S_2 we must have recourse to an annual process, using the 12 values of X_{20} resulting from the analyses for 12 months. Now the increment of angle of K_2 is about 2° per day, so that in a year it changes phase by 720° relatively to S_2. Thus the 12 values of X_{20} will show a semiannual oscillation which will be due to K_2. Another set of multipliers m_2 and m_b will give quantities which are denoted by X_{202} and X_{20b}, and these contain mainly contributions from K_2.

In a similar way, ν_2 and N_2 are isolated, and many other constituents also. The processes are thus divisible into three : (*a*) daily processes giving species of tides, (*b*) monthly processes separating groups of constituents and (*c*) annual processes separating the constituents of each group.

Similar principles apply to the diurnal and other species, the group numbers for important constituents being as follows, in continuation of Table 13.8.

TABLE 13.9

Group No.				
0	S*a*, S*sa*	K_1, S_1, P_1	S_4, SK_4	..
1	M*m*	J_1, M_1	..	..
2	MS*f*, M*f*	O_1	MS_4, MK_4	$2SM_6$
3	..	Q_1	SN_4	..
4	..	..	M_4	$2MS_6$
5	..	..	MN_4	MSN_6
6	..	..	..	M_6

The multipliers used in the monthly processes are the same for all constituents having the same group number.

This table can be verified from the table given in Art. 7.1, by multiplying the speeds by 24, and subtracting the nearest multiple of 360°, which gives the increments in phase per mean solar day. It will be found that these are approximately equal to $\pm$ 12° multiplied by the group numbers given in Tables 13.8 and 13.9.

The results will need correcting, and the designer of the method of analysis provides the proper formulæ. The corrections are simple in character. Theoretically, any function X_{pqr} will contain multiples, large, small or negligibly small, of both A and B for every constituent of the species, but by the proper choice of multipliers the A's and B's are automatically separated, so that the functions X_{pqr} contain A's, say. In Table 13.3 we have a simple illustration, in which the function X_1 contains multiples of A_1 and A_3 and the latter is eliminated by using X_3. In a similar manner, but in a much more complicated way, the designer of a method of analysis works out the final multipliers to be applied to all the functions X_{pqr} in order to isolate the required value of A.

All this is done once for all, and the computer only needs to understand the general principles as outlined above. There are, of course, many possible combinations of multipliers, but the actual choice is a matter for the exercise of much knowledge and skill and is left to the expert. It is clear that whatever system of multipliers is used in the daily, monthly and annual processes the designer can work out exactly what they will yield with any harmonic constituent. He only needs to work out the appropriate tables as in Table 13.1 for the exact speeds, and then to apply the multipliers.

13.10. The reduction of short lengths of observations

In a large number of cases it is only possible to obtain observations over a span of 15 or 29 days. In such cases it is quite impossible to separate two constituents in the same group, simply because their phases do not separate by a sufficient amount in the course of a month. Thus K_1 and P_1 cannot be separated, and the

results of analyses, carried out for the daily and monthly processes, give an apparent constituent made up of both these important constituents. In such a case we must assume the same relationship between K_1 and P_1 as exists between the corresponding equilibrium constituents. This point was discussed fully in Chapter IX in connection with the Admiralty method. The assumption is quite permissible for constituents whose speeds are nearly equal, and it is precisely in such cases where the assumption is needed. Therefore in the schedules of analysis for K_1 and P_1 a factor and phase-correction have to be evaluated, which depend upon the differences in phase between K_1 and P_1 and also upon the value of f and u for K_1. These corrections are made quite simply. Similarly it is necessary to correct the apparent constituent S_2 on account of perturbations by K_2 and T_2, N_2 is corrected for ν_2 and MS_4 is corrected for MK_4.

From short lengths of observations it is inadvisable to attempt to evaluate the smaller constituents because of the casual errors being insufficiently reduced.

When the span of the observations is only 15 days, the assumptions made when dealing with observations over a span of 29 days are exactly the same, and the principles of operation are the same, but two groups of constituents whose group numbers differ only by 1 will only separate by approximately 180° in phase during 15 days, so that the isolation of one group is much more imperfect than in the case of the span of 29 days where the phases of adjacent groups change by about 360° in 29 days. Consequently the corrections are much larger than in the case of the longer span.

The analysis of still shorter lengths of records is dependent upon more assumptions about which it is not always possible to feel great assurance. While it would be possible to obtain by direct analysis results for spans of seven days on much the same lines as for 15 days and 29 days, it may be considered as impracticable to use direct methods for spans shorter than seven days.

When only 24 or 25 hours of observations are available, then direct analyses will only determine the whole tide of any species. The methods of Art. 13.4 will give values of X_0, X_1, Y_1, X_2, Y_2, . . . but to determine by direct analysis the contributions of M_2, S_2, N_2, . . . to the functions X_2, Y_2 is impossible. Under such circumstances, all that can be reasonably done is to assume regional relationships between the principal constituents. Thus if the ratios of amplitudes of S_2 and M_2, together with the difference in the phase-lags, can be assumed to be much the same all over a region, then these quantities, if known, may be used to determine the constituents. The discussion of Art. 9.1 shows that this is normally a reasonable assumption to make. In practice, the Admiralty method automatically assumes also a relationship between N_2 and M_2.

These assumptions fail near an amphidromic point because each constituent has its own amphidromic distribution and the relations between the constituents will vary greatly near such places.

13.11. Deduction of harmonic constants

Referring back to Art. 13.2, the result of the harmonic analysis is to give the quantities A and B which specify a constituent of speed n as

$$A \cos nt + B \sin nt$$

The origin of time ($t = 0$) is chosen to suit the method of analysis, so it is arbitrary, and it is needful to express the harmonic constituent in a more suitable form. The first stage is to express it in the form

$$R \cos (nt - k)$$

as in (13.2a) with the relations of (13.3b) connecting A, B, R, k.

Since R denotes the amplitude of the constituent as obtained from the span of observations analysed, it is needful to make allowance for the nineteen-yearly variations of the constituent (see Art. 6.7) and the nodal factor, as it is called, being denoted by f, we have

$$R = fH \qquad . \quad . \quad . \quad . \quad . \quad (13.11a)$$

and H is one of the harmonic constants (see Art. 6.8).

The arbitrary time origin is inconvenient and it is desirable to adopt a system of reference, so that all constituents from all analyses, everywhere, can be rendered to the same standard of reference. At one time, in the early days of harmonic analysis, it was thought that the best reference would be to the corresponding constituent of the equilibrium tide at the place, and if this had a phase of $(V + u)$ where V is uniformly varying with speed n and u is the nodal angular correction, then by subtracting a phase-lag κ and equating the phases we have

$$nt - k = V + u - \kappa$$

and therefore

$$\kappa = V_0 + u + k \quad . \quad . \quad . \quad . \quad (13.11b)$$

where V_0 is the value of V at the origin of time ($t = 0$). The phase-lag κ is one of the harmonic constants (see Art. 6.8).

It was thought that the value of κ, being related to the local tide-generating forces, might have a dynamical significance, but it was found to be a cumbrous procedure to compute V for all places on the earth, and more convenient to tabulate it for the Greenwich meridian, and a simple and logical outcome was to treat the observations as though they had been taken at Greenwich (see Art. 7.3). The observations taken in standard time are now treated as though they were taken at Greenwich, and the phase-lag on the Greenwich equilibrium constituent is defined by g, and related to κ by

$$g = \kappa + jL - nS \quad . \quad . \quad . \quad . \quad . \quad (13.11c)$$

where j is the species number (0 for long periods, 1 for diurnals, etc.)
n is the speed in degrees per mean solar hour
L is the longitude of the place in degrees west of Greenwich
S is the longitude of the time meridian in hours west of Greenwich.

(For a proof of this formula see Art. 7.3.)

One advantage of this method is that all places in a time-zone are related to one another so that the differences in g of M_2, say, divided by the speed (29° for M_2) give the difference in hours of mean solar time between the mean high water times, and another advantage is that if predictions are made for all places as though they were on the Greenwich meridian the predictions are given automatically in standard time.

The phase-lag g is now always used in Admiralty tidal practice, but as the older publications give κ, which constant is also still used by some authorities, the formula (13.11c) is very important.

13.12. The Admiralty Semi-Graphic Method of Harmonic Tidal Analysis

Although methods of analysis similar to those described in this chapter are still used by the Liverpool Tidal Institute, a new semi-graphic method has been developed in the Admiralty for the analyses of periods of one month. This method is fully described in Admiralty Tidal Handbook No. 1, The Admiralty Semi-Graphic Method of Harmonic Tidal Analysis.

CHAPTER XIV

THE PREDICTION OF TIDES

14.1. Non-harmonic methods

THOUGH purely non-harmonic methods of prediction are now almost obsolete, it would be unfair to ignore them entirely in this Manual, particularly as the Admiralty method is essentially non-harmonic in conception though it utilises harmonic constants.

The simplest method of computing approximate predictions of tides, in British waters and in some other parts of the world, is to add to the time of transit of the moon a constant interval appropriate to the place for which times of high and low water were required. The heights of high water are obtained by estimation between the heights of H.W. springs and H.W. neaps, though sometimes only H.W. springs is given. This method has been largely practised and is still used in pocket-diaries, and for tides at seaside resorts. It is entirely unsuitable for modern needs of navigators.

More elaborate methods were utilised in computing tide-tables, but the essential principles were much the same, though corrections for parallax were included. The Equation method is probably the most elaborate of these methods and has been used extensively in predicting tides for the Admiralty Tide Tables. It can be said to have exhausted the possibilities of non-harmonic methods for first-class predictions. It yields very good results where either the semidiurnal tide or the diurnal tide is greatly predominant, but mixed tides in which semidiurnal and diurnal tides are both prominent have proved intractable by these methods which have therefore gradually fallen into disuse.

A non-harmonic method which yet finds satisfactory use in certain cases uses differences between the high and low water times (or heights) at neighbouring places. The principles of this method have been discussed in Chapter IX, and it is only applicable if two places are so near together that the relations between constituents of the tides are much the same at the two places. Generally speaking it is not applicable with any confidence if the time-differences vary greatly from springs to neaps or if the differences for low water are much different from those for high water or if alternate tides differ appreciably, as these discrepancies indicate substantial alterations in the first case in the inter-relations of the tidal constituents of the predominant species, in the relative ranges of semidiurnal and quarter-diurnal tides in the second case, and in the relative ranges of diurnal and semidiurnal constituents in the third case.

The method therefore needs to be used with caution, but under suitable conditions it can yield very satisfactory predictions with greater ease and at less cost than by the direct methods.

14.2. Computation by harmonic methods

It was the great difficulty of dealing with mixed tides which led to the development of the harmonic method by Lord Kelvin, but if the harmonic constituents had needed to be synthesised by direct computation in order to produce predictions the method would probably have had a very limited use, though many predictions have been made by direct methods of computation, in which the principal harmonic constituents have been computed hour by hour for the whole year and the results summed in order to give the heights of tide hour by hour. From these hourly results the heights and times of high and lower water can be computed. The process can be shortened if only high and low waters are required, by evaluating the constituents at times near high water, using mean lunar time. These computations are very laborious

and even the approximate methods published by the Admiralty in the year 1927 for use over a span of 24 hours, with many ingenious helps for computation, involved a degree of computation which rendered the method somewhat unpopular, though it was at the time a necessary evil in the absence of a shorter method for predicting mixed tides.

14.3. Elementary principle of the tide-predicting machines

The evolution of the harmonic method has been very largely dependent upon the possibilities of using it to produce predictions of tides by mechanical means. To Lord Kelvin belongs the credit of first suggesting the use of a machine and for the construction of models which demonstrated their possibilities. His assistant, Mr. Edward Roberts, shares the credit with Lord Kelvin for developing the machines and putting them into practical form.

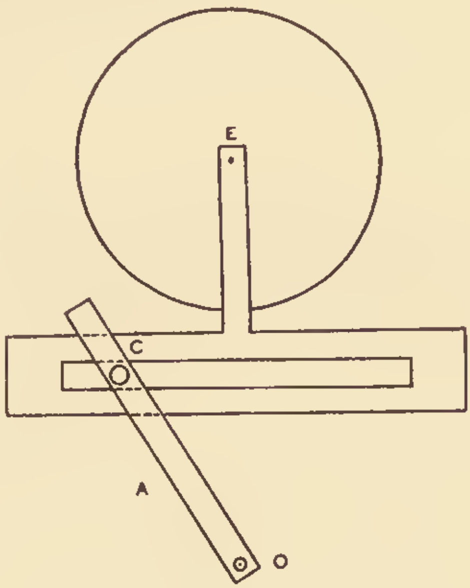

FIG. 14.1. Generation of harmonic motion.

The principle of the machine is simple, its object being firstly to generate harmonic motions in a vertical line, and secondly to sum the motions by mechanical means. Harmonic motion is easily generated by the revolution of a crank revolving round a centre at a uniform rate. In Fig. 14.1 let the crank revolve round a centre O; let C be a pin, fixed in the crank, sliding in the horizontal slot of the T-piece, which, with the pulley E, is free to move in the vertical direction only. Then the elevation of the slot above the centre of rotation is equal to OC cos COE. Hence the distance OC must be made to represent on a suitable scale the amplitude (R) of the constituent, and the rate of revolution of the crank must be proportional to the speed of the constituent, and the initial angle at $t = 0$ (say, at zero bour on January 1st for the required year) must be capable of being set. Then, as the crank revolves, the pulley E connected to the T-piece will rise and fall in a harmonic motion proportional to that of the harmonic constituent.

A large number of such units can be very compactly geared to a main shaft in such a way that their rates of revolution are directly proportional to the speeds of the constituents. This is simply a matter of the number of teeth in the gear wheels (see Art. 14.4).

The solution of the problem of summing the motions is shown in Fig. 14.2. The pulley wheels, E, F, G, . . . are all placed in the same plane and a wire, fixed

at one end, X, passes round the pulleys until the free end hangs vertically under the weight of a pen carriage P. Then as E rises the pen is drawn upwards, but as F rises the pen is let down. In order to make the two conspire in increasing the movement of the pen, such as would be required for the conspiring of two constituents, it is necessary so to arrange the angular scales that a downward movement of F corresponds to a rise of elevation due to the constituent, and so on for every pulley in the lower loops of the wire.

Thus, since the wire is fixed at one end, the vertical movements of the pulleys are summed automatically by the wire, and if the pen carriage maintains the pen

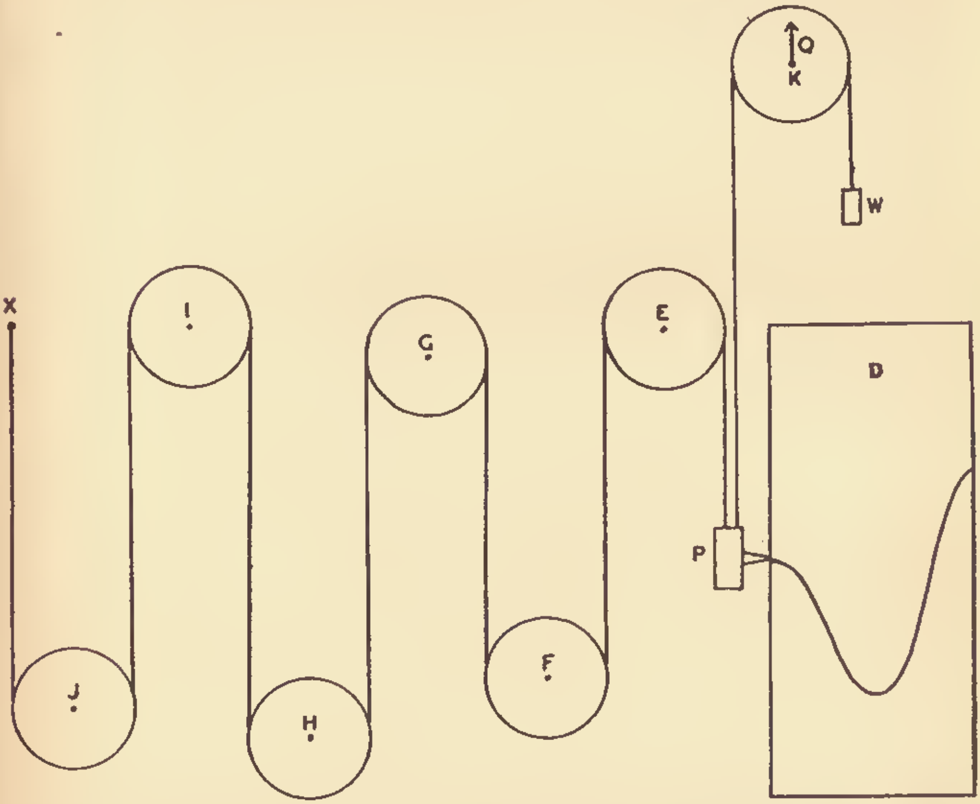

FIG. 14.2. Use of wire in summing the harmonic motions.

in contact with paper on a drum D suitably geared to the main shaft, a curve will be drawn as D revolves with the shaft, and the pen sums the movements of the pulleys. This curve can be read off as required.

Alternatively, if there is a separate connection to the wire, say from the pen carriage, over a pulley K, the wire being kept taut by a weight W, then the readings of a scale on the pulley can be read off against a fixed pointer Q.

14.4. Gears and gear-ratios for the speeds of constituents

The mechanical realisation of a machine as visualised in the preceding article offers problems mainly of an engineering character, but there are certain matters which call for examination, particularly in connection with the possibility of ensuring that the crank wheels for all the constituents will revolve at the correct relative speeds without serious gain or loss in angle. Engineers are familiar with the principles of computing the number of teeth in gear wheels for any required ratio, but it is of general and permanent interest to explain briefly how this is effected in the

particular problems of the tide-predicting machine. There are 8760 hours in a year of 365 days, and therefore for a semidiurnal constituent (S_2, for precision) the increase in phase amounts to 262,800 degrees of angle in the year. For an error of less than $0^\circ{\cdot}3$ the gears must be accurate to one part in a million!

The main problem is that of the speeds of the constituents relative to one another and we shall therefore take as a basis the speed of the solar diurnal constituent S_1, to which all others will be related through appropriate ratios. This leaves as a "disposable quantity" the relation between S_1 and the main shaft, so that if necessary the gear ratio of an important constituent such as M_2 may be improved by using a suitable relation between S_1 and the shaft.

The gear ratios for all solar constituents will be simple numbers; thus the ratio between S_2 and S_1 is as 2 to 1. Any convenient number of teeth on the gear wheels may then be taken so long as the correct ratio is kept and there is no appreciable backlash. Generally speaking the number of teeth should be as great as is mechanically possible.

Now consider the constituent M_2, whose speed is $28^\circ{\cdot}984104$ per mean solar hour, which is 1·9322736 times the speed of S_1. We have to find a ratio of numbers of teeth which will be as accurately as possible equal to this number. The approach to an acceptable ratio is readily effected by the following method, which is much simpler than methods in common use. The first step is to obtain two ratios correct to within, say, 5 per cent., and this is readily effected on a slide rule by merely setting the ratio of the scale and observing the cases where integral numbers come opposite to one another on the rule. Alternatively the ratio may be added to itself continuously until the result is approximately an integer. By these means we get the two approximations

$$\frac{a}{b}=\frac{2}{1}, \text{ or accurately, } \frac{a}{b}=\frac{2-0{\cdot}067726}{1} \quad . \quad . \quad . \quad (14.4a)$$

and

$$\frac{c}{d}=\frac{27}{14}, \text{ or accurately, } \frac{c}{d}=\frac{27+0{\cdot}051830}{14} \quad . \quad . \quad (14.4b)$$

Since the accurate values of a/b, c/d are equal to the ratio under consideration, and therefore equal to one another, then each is equal to

$$\frac{mc+a}{md+b}$$

where m is any multiplier. Hence we see that a better ratio is obtained by taking $m = 1$, and thus we obtain

$$\frac{e}{f}=\frac{c+a}{d+b}=\frac{29-0{\cdot}015896}{15} \quad . \quad . \quad . \quad . \quad (14.4c)$$

(This, of course, was obvious in this particular case, but we adopted a general method rather than make use of a fortuitous simplicity.)

On considering (14.4b) and (14.4c) we note that the number 0·051830 is roughly three times the number 0·015896, whence we can obtain a better approximation to the ratio in the form

$$\frac{g}{h}=\frac{3e+c}{3f+d}=\frac{114+0{\cdot}004142}{59} \quad . \quad . \quad . \quad (14.4d)$$

and proceeding in this manner we successively obtain

$$\frac{i}{j}=\frac{4g+e}{4h+f}=\frac{485+0{\cdot}000672}{251} \quad . \quad . \quad . \quad (14.4e)$$

$$\frac{k}{l}=\frac{6i-g}{6j-h}=\frac{2796-0{\cdot}000110}{1447} \quad . \quad . \quad . \quad (14.4f)$$

We have now arrived at approximations of the right order of accuracy, for the error of taking the ratio 485/251 is proportional to 0·000672 in 485, or a little over one part in a million. The actual error in the increment of phase per year amounts to $0^\circ{\cdot}4$. The ratio 2796/1447 is more accurate still, with an error of $0^\circ{\cdot}01$ per year.

Before proceeding further we must consider the mechanical application of these formulæ. In Fig. 14.3 let two wheels on the driving shaft have teeth denoted by s and m for the S_1 and M_2 components and let these drive the corresponding crank wheels whose numbers of teeth are denoted by s' and m'. Then one revolution of the shaft will give s/s' revolutions of the S_1 crank wheel and m/m' revolutions of the M_2 crank wheel so that one revolution of the S_1 crank corresponds to $\frac{m}{m'} \times \frac{s'}{s}$ revolutions of the M_2 crank wheel.

If we choose the ratio $s'/s = 1$ so that the S_1 component is driven at the same angular rate as the shaft then the gear ratio for M_2 to S_1 is denoted simply by m/m'. Now the practical limits for the number of teeth and size of wheels makes the gear ratio (14.4f) unusable, and thus we become limited to the choice of (14.4e) for the gear ratio for M_2, when s'/s is specified in this way.

If, however, we do not discard the freedom of choice of the rate of revolution of the shaft (relative to that of the S_1 crank wheel) then we can use larger numbers

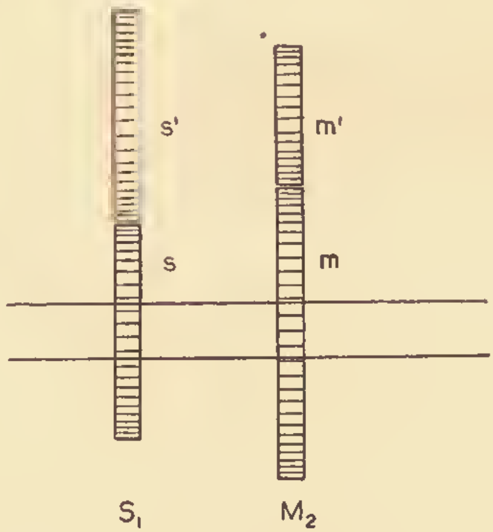

FIG. 14.3. Gear wheels for S_1 and M_2.

for the gear ratios, with, of course, much greater accuracy, but it is evident that the numerator and denominator must each split up into factors. Now in the case of the gear ratio (14.4f) this is not possible, for 1447 is a prime number. We must therefore proceed a stage further with our calculations, not that we really need greater accuracy than is given by (14.4f) but simply in order to obtain factorisable quantities. We therefore have an element of choice in our continuation of (14.4f) and (14.4e) and s obtain either

$$\frac{m}{n} = \frac{5k+i}{5l+j} = \frac{14465 + 0{\cdot}000122}{7486} \quad . \quad . \quad . \quad (14\ 4g)$$

or

$$\frac{o}{p} = \frac{6k+i}{6l+j} = \frac{17261 + 0{\cdot}000012}{8933} \quad . \quad . \quad . \quad (14.4h)$$

the errors of phase-increment per year being respectively $0°{\cdot}002$ and $0°{\cdot}0002$. The errors in each case are negligible, but since 8933 is a prime number we cannot use the latter, and exceedingly accurate, formula, but a very accurate and suitable relation between the rates of revolution of the M_2 and S_1 crank wheels is given by the gear ratio

$$\frac{14465}{7486} = \frac{263 \times 55}{197 \times 38} = \frac{263 \times (55i)}{197 \times (38i)} \quad . \quad . \quad . \quad (14.4i)$$

where i is an integer, and this must be related to the expression

$$\frac{m}{m'} \times \frac{s'}{s} \quad \ldots \ldots \quad (14.4j)$$

obtained above.

We give the alternative expression involving $55i$ and $38i$ because gear wheels with only 55 or 38 teeth may not be suitable mechanically. We note that there is still an element of choice regarding s' and s; thus we can take s equal either to 197 or to $38i$ while s' can be taken as either 263 or $55i$, but this choice when once made for the most important constituent restricts the freedom of choice for other constituents, unless they are driven from secondary shafts with suitable gears between them and the main shaft.

We have shown therefore that the important problems of gear ratios are capable of very accurate solution and we proceed to refer to the application of the principles

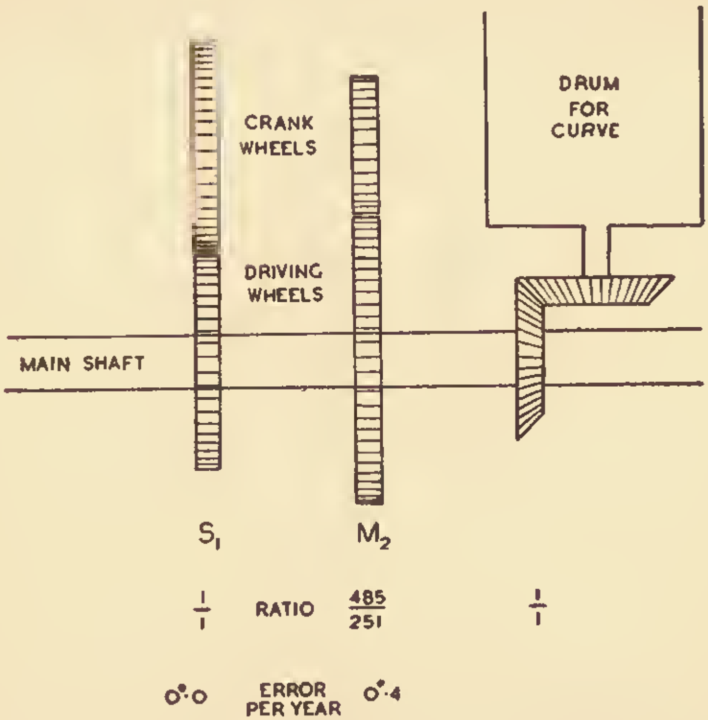

FIG. 14.4. Arrangement of gear wheels in Kelvin Machine.

in the case of the two machines owned by the Liverpool Observatory and Tidal Institute and principally used for the Admiralty predictions.

The machine made by Kelvin, Bottomley and Baird, like all the other machines made by them, utilises very simple gears as illustrated in Fig. 14.4 for S_1 and M_2. It will be noted that there is a 1/1 ratio between the ratios of revolution of the shaft and the S_1 crank, so that the best possible ratio for M_2 under these circumstances is that already considered (485/251) with an error of 0°·4 per year. This error is just within permissible limits. The other machine, designed by Mr. E. Roberts many years ago, and made by Légé & Co., has much more elaborate gears (Fig. 14.5) using an intermediate worm gear W on the shaft of a wheel C between the crank wheel A and the wheel B located on the main shaft. The teeth used on the set of wheels for S_1 and M_2 are as follows :—

	A	B	C
S_1 . .	263	180	110
M_2 . .	197	180	76

It is evident that these come from the formula (14.4i), and that owing to the use of multiple gears there is still an element of choice as regards the teeth on the wheel denoted by B.

The machine used by the United States Coast and Geodetic Survey is also a multiple gear machine and the ratio of speeds of M_2 to S_1 is:

$$2 \times \frac{103}{74} \times \frac{59}{85} = \frac{6077}{3145}$$

which is readily obtained from (14.4f) and (14.4e) as

$$\frac{2k + i}{2l + j}$$

There is one other point which may be mentioned in connection with the gear system. When the cranks of the Kelvin machine are being set to their initial angles there is of necessity a tendency through friction to rotate the main shaft also, and

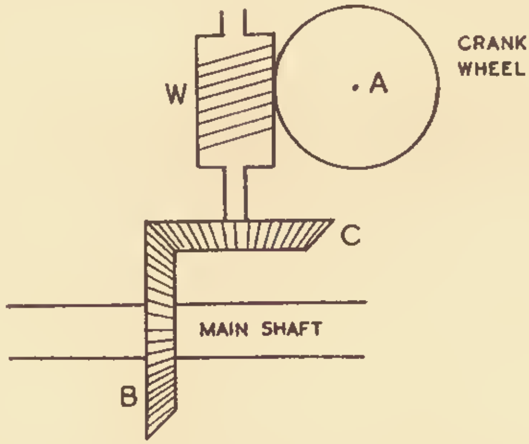

FIG. 14.5. Arrangement of gear wheels in Légé Machine.

care needs to be exercised in setting the machine. This tendency is counteracted in the Légé machine through the presence of the intermediate worm gear.

As regards the mechanical or engineering problems of laying out the gears we need say nothing. The simple gears of the Kelvin type of machine are very quiet in action and there are fewer problems of maintenance and repair because the gears are accessible on an open shaft, whereas the Légé system of gears needs to be enclosed between steel plates and the meshing of the various wheels needs very great care and skill whenever the machine requires adjustment.

14.5. The harmonic motion

The revolution of the crank wheels must be transformed to harmonic motion in a vertical line, and the general principle of doing this was explained in Art. 14.3, but there are certain considerations which are worth attention. Since a machine of a reasonable size necessarily involves a small scale (one or two centimetres per foot of tide) it is of importance to ensure that great mechanical accuracy is maintained in the harmonic motions.

As the crank revolves, the pin denoted by C in Fig. 14.1 slides to and fro in the slot of the "oross-head" or T-piece; true harmonic motion only results when the edges of the slots are perfectly straight and it is necessary that this shall be maintained even if there is wear. Now if the movement is small the to and fro movement of the pin will result in the upper edge of the slot being worn to the shape given in Fig. 14.6 and the motion is then not truly harmonic; in fact, the errors so caused may be quite appreciable. In order to counteract this tendency the pin in

the crank should rotate in a " slipper-block " (Fig. 14.7). By this artifice the pressure exerted by the pin on the edge of the slot is spread over a much greater area and the edges of the slot wear uniformly.

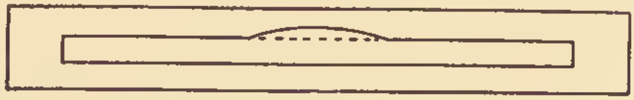

FIG. 14.6. Wear in slot affecting harmonic motions.

FIG. 14.7. Slipper block used to minimise effects of wear.

14.6. Counterpoises

Another important consideration, which must not at any time be overlooked, is that of ensuring contact between the slipper block and one edge only of the slot in the T-piece, for if the block can move from edge to edge of the slot there will be lost motion which will give appreciable errors, seeing that the scale of motion is small.

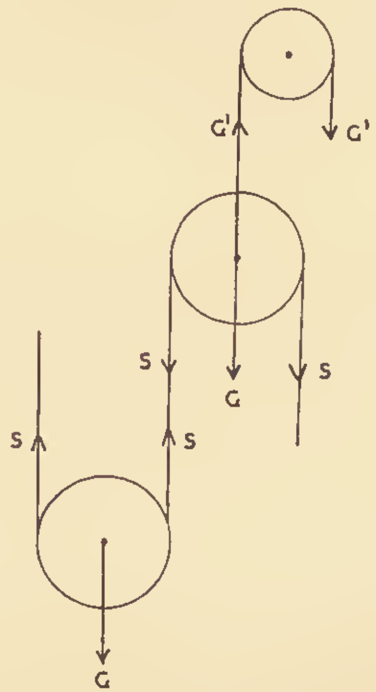

FIG. 14.8. Design of counterpoise.

The pulleys, as in Fig. 14.2, are arranged in two rows, and it is evident that in the lower row the tension in the wire will tend to counteract the weight of the assembly of pulley and T-piece, while in the upper row the tension in the wire will reinforce the weight of the pulley and T-piece, so that there will be an excessive pressure upon the slipper-block. It is desirable therefore to attach to each upper pulley a

counterpoise weight by means of fine wire or string which passes over a pulley. The system is illustrated in Fig. 14.8.

Let the tension in the wire be denoted by S and let the weight of the pulley and T-piece be denoted by G, while the counterpoise weight in the case of the upper pulley is denoted by G′. Then the downward pressure exerted on the slipper block is equal to $G - 2S$ in the case of the lower pulley and equal to $G + 2S - G'$ in the case of the upper pulley.

In order to give equality of pressures so that all the slots in the T-pieces will wear equally, we must make

$$G - 2S = G + 2S - G' \quad . \quad . \quad . \quad . \quad (14.6a)$$

so that the counterpoise weight is given by

$$G' = 4S \quad . \quad . \quad . \quad . \quad . \quad (14.6b)$$

Alternatively, if we make the tension in the wire such that 2S is greater than G then the contact between slipper-block and T-piece will take place on the upper edge of the slot in the case of the lower pulleys and we can contrive the contact to take place at the upper edge of the slot in the T-piece in the case of the upper pulley; therefore, as the resultant pressures are opposite in direction, then equality of magnitude will yield

$$G - 2S = -(G + 2S - G') \quad . \quad . \quad . \quad . \quad (14.6c)$$

and thence

$$G' = 2G \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.6d)$$

Whichever method is chosen there is one uncertain factor which may easily be overlooked. So far we have considered equality of magnitudes of pressures, but to ensure these pressures being maintained in one direction in any particular case is by no means a simple matter. The relations (14.6b) and (14.6d) are based upon assuming $G > 2S$ in the former case and $G < 2S$ in the latter case. It may seem an obvious solution to choose the latter relation, for we can make S as big as we choose by adding weights to the pen carriage, with the advantage that the wire is then kept perfectly taut, but the solution of the problem is not effected so simply, for the tension in the wire is not constant, as we shall proceed to show.

14.7. Effects of friction in the pulleys

The tension on either side of a pulley is not the same unless there is absolutely no friction at the axle of the pulley. The general theory of friction shows that there will be the same percentage loss of tension at each pulley. Consider a single pulley and let the tensions in the wire on the left and right sides of it be denoted by S_1 and S_2 respectively. Now if the pulley is turning clockwise so that the wire on the right is moving down then the friction at the axle will resist the motion so that the tension S_2 is then greater than S_1 by the amount of the frictional force. If, however, the movement is reversed then S_1 will be greater than S_2 by the same amount of frictional force.

Now consider a system of N pulleys and let only the one nearest to the fixed end of the wire be acting. Then if the motion is such that the pen carriage is falling, it is the weight of the pen carriage (W) which is the effective force and therefore the tension in the wire will decrease from the free end to the fixed end. If, however, the pen carriage is being pulled up, then the tension at the fixed end is greater than the tension at the free end of the wire.

Let the percentage drop of the wire due to friction be $x\%$ at each pulley, then the tension near the fixed end of the wire will vary from $W\left(1 - \frac{Nx}{100}\right)$ to $W\left(1 + \frac{Nx}{100}\right)$ whenever the direction of motion changes, under the circumstances considered (viz., that all the pulleys are reversing their directions together on account of the motion being controlled by the component nearest to the fixed end of the wire). If the change of tension is about 1%, so that $x = 1$, and if N is 20 then the change-over in tension at the fixed end of the wire is as much as 40% of the tension at the free end.

Such changes can readily be measured, and experience shows that the changes in tension are of the above order. Such large changes will complicate the calculations for the counterpoise weights discussed in the preceding article, and experiments may be needed to verify that under no conditions will the slipper-blocks cross the slots.

But there is another evil arising out of friction at the pulleys. A wire running round a system of 20 or even 40 pulleys will be very long, of the order of 20 or 40 ft., say. Under tension the wire stretches, but this does not matter if the variation in tension is negligible. In practice, unless proper measures are taken, the effect of variable tension due to friction is much more important than that due to the slipper-blocks crossing from one edge of the slot to the other. An obvious solution is to use a strong wire which will be almost immune from stretching under the tensions experienced, but such a wire will not be very flexible and will not " lie " properly from one pulley to another. A steel wire will resist stretching, but because of its

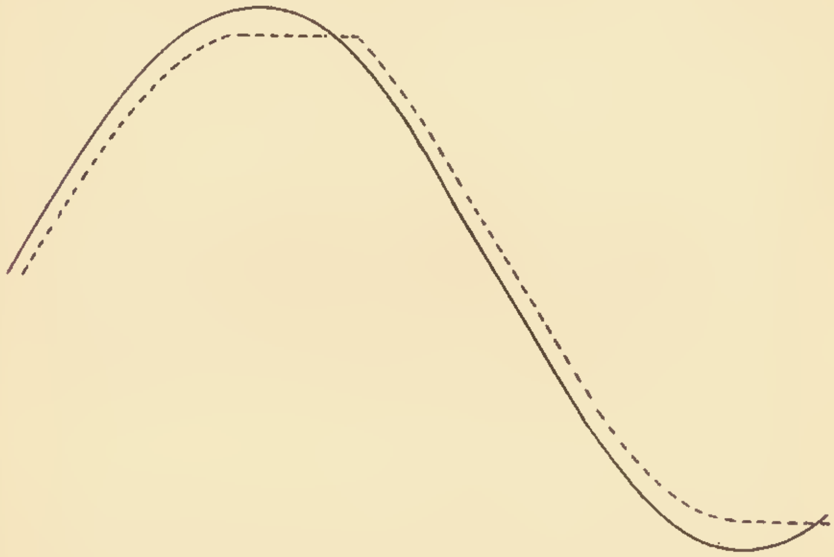

FIG. 14.9. Effect of friction in axles of pulleys.

springy character it will go round from one pulley to another in an exaggerated S-form, and any variation in tension will affect the curvature of the wire and therefore the movement of the pen-carriage.

Experiments with the wire of a tide-predicting machine have shown that the springiness and stretch of the wire each yield much the same result on the motion of the pen ; a soft copper wire will lie perfectly from one pulley to another but will be easily stretched, while a steel wire or tungsten wire will resist stretching but will also refuse to lie properly between the pulleys. In each case the effect is to reduce the range of tide and instead of getting the curve shown by the full line in Fig. 14.9 we get the effect shown on an exaggerated scale by the broken line. When the curve is rising the pen lags behind by a constant amount (for the frictional forces only depend upon the direction of the motion and not upon the duration) until high water is reached. Then, as the frictional forces change over at the reversal of motion, the pen will not move until it lags behind its proper position by an amount equal to that experienced on the rising tide. We thus get flat tops to the curves.

The magnitude of the error will increase with the number of pulleys, other things being equal. A 30-component machine may give an error of several inches of tide for a large tide which is to be predicted on a small scale of 1 cm. or even 0·5 cm. per foot, but the error is reduced if the scale is made larger, for the frictional effect, as has been pointed out above, does not depend upon the amplitude of motion.

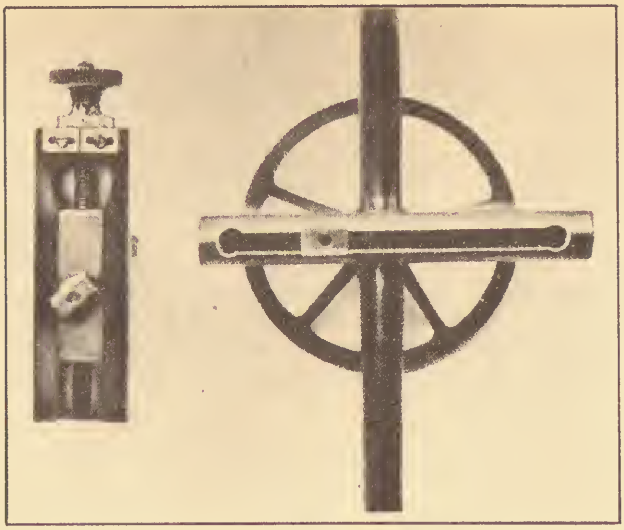

Fig. 14.10. Lége component.

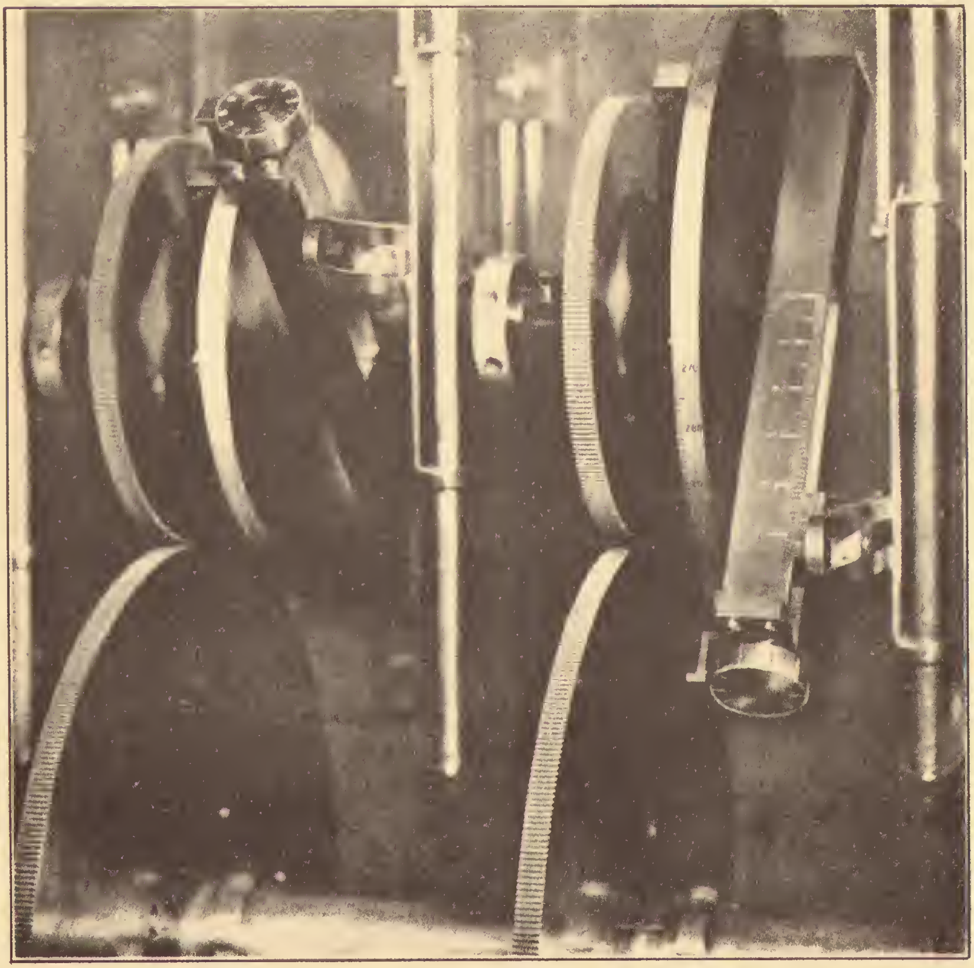

Fig. 14.11. Kelvin component.

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14.8. Reduction of frictional effects

A chronometer chain made of very small links is one alternative to the use of a wire, but suffers from other disadvantages. It is in itself rather heavy, its performance depends upon the performance of each link and the problem of wear in the pivots introduces considerations of the effective tensions to ensure that there is unvarying mechanical contact. In many ways a chain offers more sources of trouble than a wire.

Experiments showed that wires with circular cross-sections offered little or no choice as between one material or another, but a consideration of the mathematical laws regarding spring flexures showed that a flat wire tape would be free from errors due to imperfect flexure, and the area of the cross-section could be made large enough by taking the tape as wide as possible so that it would resist appreciable stretching. Most modern machines now use a tape made of pure nickel, which is very tough and thus not easily broken, but has a high resistance to stretching and is very flexible. The tapes in use are about a millimetre wide and a fifth or tenth of a millimetre thick. Such tapes reduce the frictional errors to about one-fourth of the errors associated with wires of a circular cross-section.

The use of these wires, of course, is only a palliative, and it is better to reduce the error at the source. The best way of doing this is to provide each pulley with ball bearings. It is only in recent years that ball races of a suitably small size have been available, but ball races of remarkably small sizes can now be obtained. The machines constructed in recent years by Kelvin, Bottomley and Baird, acting with the advice of the Liverpool Observatory and Tidal Institute, who have initiated these improvements, all use ball races. The Kelvin machine owned by the Tidal Institute, of course, is fitted with ball bearings, and the older machine, also now owned by the Institute, made 40 years ago by Légé to the design of Mr. E. Roberts, has also been modernised by Chadburn's (Ship) Telegraph Co.

The errors of these two machines, regarded as computing machines, are now less than a fraction of an inch for any range of tide.

It is desirable to point out, however, that even ball races may be a source of trouble unless they are kept perfectly clean and free from grit. In high-speed practice the inner case of a ball race is fitted tightly in its bearing, but this practice should not be followed for tide-predicting machines. The ball race should act as a free (but not slack) pivot in the event of the race seizing up due to grit.

14.9. Description of machine components

The mechanism for producing the harmonic motions is quite simple, though great exactitude is necessary in manufacture. Fig. 14.10 shows a picture of the Légé type of component and Fig. 14.11 shows the Kelvin components. There are slight differences between them but one description will suffice. The crank pin is set at a predetermined amount by turning the head of a setting screw, and the pin itself is set in a long hlock which slides in a suitable groove. The exact amount by which the pin is moved away from the centre is shown by the reading of an engraved line on the sliding block against a scale on the edge of the groove. This is engraved in millimetres and numbered in centimetres. The screw thread has a pitch of one millimetre so that one revolution of the head of the screw increases the amplitude of setting by one millimetre, and this screw-head is also engraved in tenths of a turn for reading against a line engraved on the mounting. By this device the number of complete turns is read on the side scale and the fraction of a turn, estimated to the nearest hundredth, read on the head of the screw. Thus, nominally, the crank pin can be set to one-thousandth part of a centimetre, but this accuracy can only be achieved in practice by making all settings in one way so as to avoid errors due to backlash. In practice the final setting, whether for zero or any other amplitude, is accomplished by a backward turn of the screw. Obviously there must be sufficient friction in the various parts to ensure that this position is maintained. Provision is made for altering the setting marks in tbe event of wear taking place.

This assembly is rigidly connected to a disc engraved on its periphery with a

scale of degrees of angle, in the case of the Kelvin machine, and this disc can be freed from the gear wheel by means of a frictionally operated clutch, so that the angular settings for the phases at the commencement of the predictions can be made against an engraved line on the mounting. In the case of the Légé machine the crank assembly is rigidly connected by a strong shaft to a pointer which moves over a scale of degrees on the rear plate of the machine, and the clutch operates on this shaft between the front and rear plates. Provision is made for ensuring exactness of angles.

The angular relations of crank and angle-disc are adjusted and verified from time to time, say once a year, by setting zero amplitude on a component and noting the reading of a micrometer attached to the component, then by setting large amplitudes and testing the readings of the micrometer, which has not been moved in the meantime, at settings of 90° and 270°. There should be no variation from the setting obtained with zero amplitude. The same tests, of course, also verify whether the zero is correct, for with a nominal zero amplitude there will be no movement on the micrometer as the component is revolved. At the Tidal Institute these tests and many others are made at least once a year.

14.10. General description of Kelvin machine

The Kelvin machine belonging to the Tidal Institute was constructed in the year 1924 and is shown in Fig. 14.12 in its teak case. The machine is driven by a small electrical motor, but it can be adjusted or even driven by hand. The 26 components are laid out in two parallel rows on the two sides of a strong steel plate, and the lay-out of parts is very simple. The gears are driven from the main shafts, and the front shaft also drives a drum for the record. This is seen on the right, and the two storing reels, one for unused paper and one for the record, are also clearly seen. The drum is provided at both ends with small sharp needles which serve to engage the roll of paper and draw it from one drum to another and also to give perforations which denote the time. Around the upper edge of the drum is an engraved time-scale.

The pen-carriage slides on rollers between two guide rails and the pen can be lifted by a spring-clamp from the paper. Just above the drum is a dial graduated in days of the year on which a revolving arm indicates the progress of the running of the machine. Above this large dial is a smaller dial on a disc running in roller bearings; this indicates the height of the tide, and it is actuated by a wire attached to the pen-carriage, this wire being kept taut by a counterpoise to the right of the dial. The height is read against a pointer, which is set to give the mean sea level when all the components are brought to zero positions.

An additional time scale on white celluloid has been fixed, since the photograph was taken, on the gear wheel on the shaft nearest to the drum.

The pillar in front of the roll of paper carries two pens, one of which gives mean sea level and the lower one gives the datum of the predictions.

Fig. 14.13 shows the fixed end of the wire and the details of the mounting of the pulleys in relation to the T-piece and driving mechanism.

14.11. General description of the Légé or Roberts machine

This machine (Fig. 14.14), designed by Mr. E. Roberts and made by Légé & Co., was put into use about the year 1908. The principles of action are exactly the same as those of the Kelvin type of machine, but it is more robust, and the moving parts for the harmonic motion are made in steel instead of brass. It was purchased by the Tidal Institute in the year 1929, and many improvements and additions have since been made. The lay-out of parts is more complicated than that of the Kelvin machine, and more gears are used, as is mentioned in Art. 14.4, so that the gear ratios are very exact, but this has entailed a separation of parts so that all the amplitudes are set on the front plate of the machine and the angles on two rear plates which are so designed that all the clutches on the shafts passing between the front and rear plates can be operated by hand.

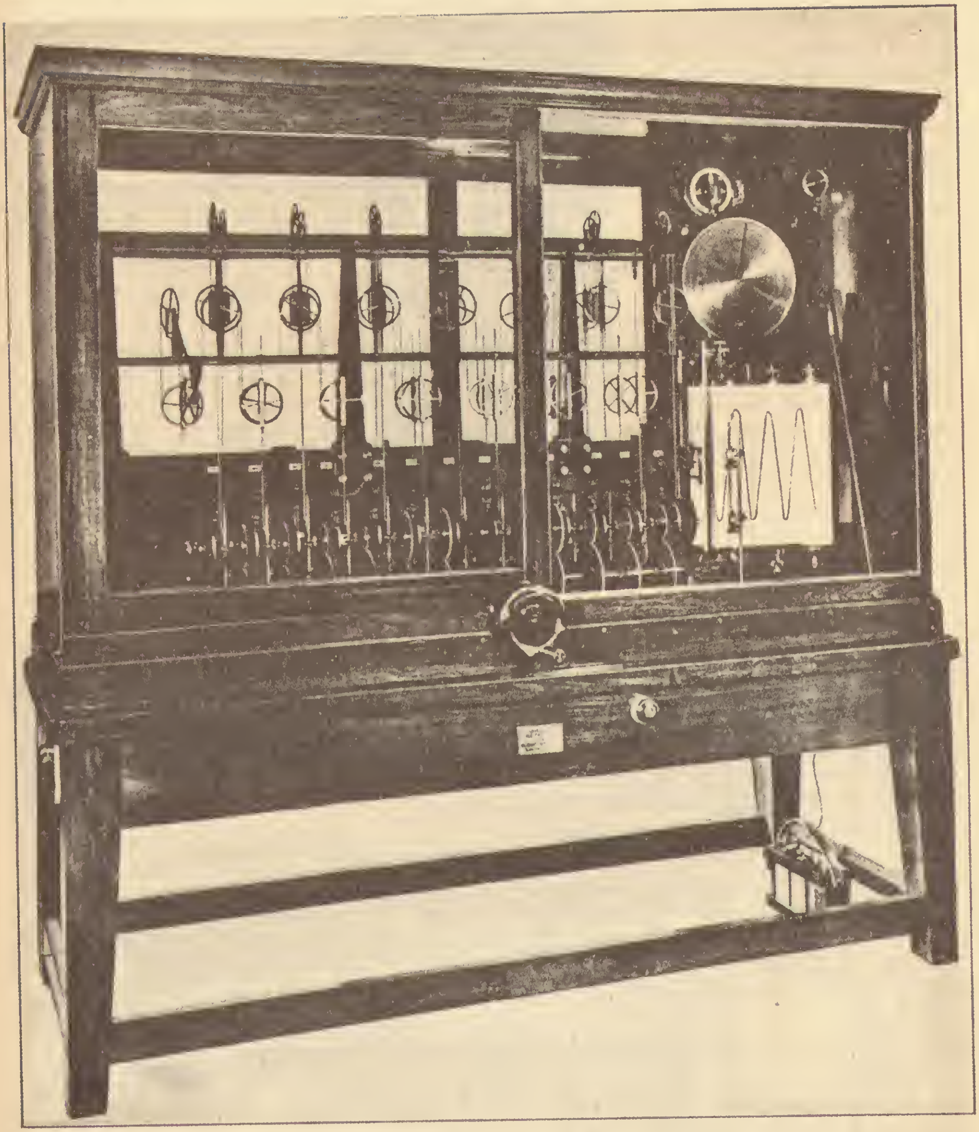

FIG. 14.12. Kelvin Machine (Tidal Institute).

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FIG. 14.13. Details of Kelvin Machine.

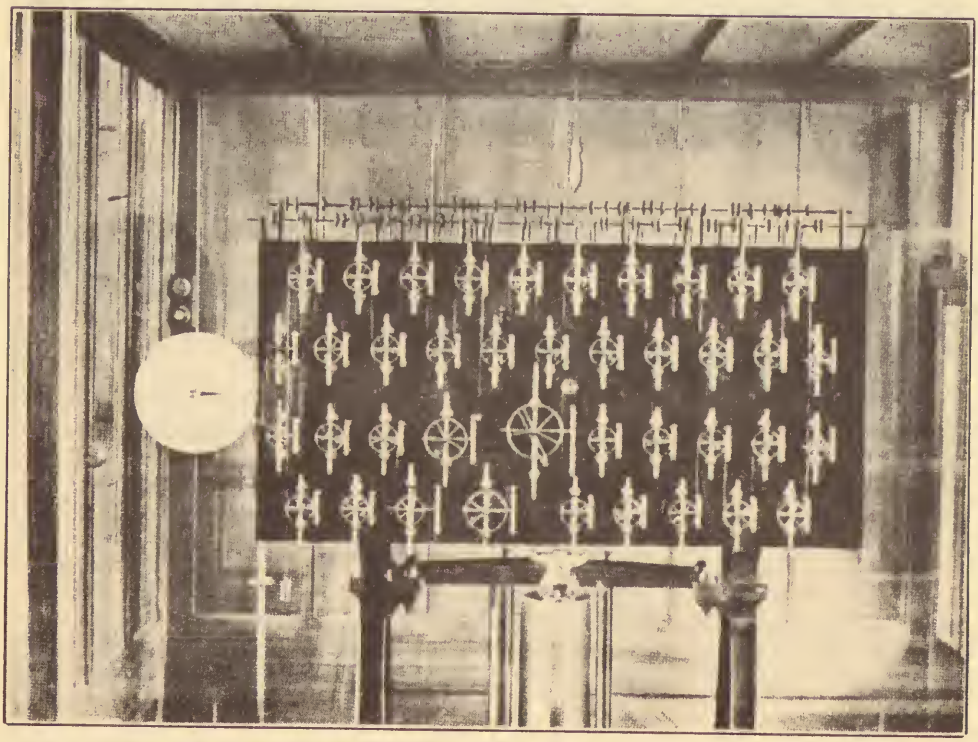

FIG. 14.14. Légé Machine (Tidal Institute).

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Instead of the principal scales of the amplitude settings being on the mounting for the setting screw they are placed on vertical strips of metal at the sides of the components, so that the approximate settings are made with the cranks in the vertical position. Each head is provided with a disc and zero-mark for fine settings of amplitude (Fig. 14.10). The separately mounted scales are convenient in that the zero positions are readily adjusted by using the slotted holes in the scales and a similar convenience is attached to the scales of angles on the rear plates, as they are readily rotated into the correct positions, so that the machine is more easily corrected for wear than is the Kelvin machine.

The T-piece for the harmonic motion is extremely strong, being made out of two steel tubes welded at right angles (Fig. 14.10).

There are four shafts, each made up in two portions for the four systems of pulleys and these are driven by a vertical shaft at the left of the machine. Very full provision is made for adjustments for meshing the gears which, however, needs to be very carefully made, and verified from time to time. A fifth shaft under the strong iron framework drives the drums.

The original machine has been greatly improved since it was purchased by the Tidal Institute, by making the grooves of the pulleys with flat bottoms to take a nickel tape, and recently by the addition of ball bearings. An engraved time-scale has been placed on the main drum and a separate dial for the indication of heights placed beneath the front plate and the drum, while the full provision for 40 components has been completed by the addition of 10 components to those for which gears had already been provided.

The accuracy of performance of this machine is now equal to that of the machine described in the preceding article, and, of course, the larger number of components adds greatly to its usefulness.

14.12. Outline of procedure in using the machines

When predictions are required for a place the harmonic constants from all available analyses are collected and a card is prepared giving the values of the harmonic constants H and g for all constituents in order of the rows of components of the machine to be used.

For each year a similarly arranged card is prepared giving the values of $V + u$ and f (see Art. 13.11) for the zero hours on the first days of January, April, October, and also January of the following year. The variations in u and f are not very great, but the Tidal Institute procedure differs from that frequently adopted, in that the values of u and f are only regarded as constant for half a year rather than a whole year.

Since a harmonic constituent can be written as

$$f\mathrm{H} \cos (nt + a) \qquad (14.12a)$$

then the curve expressing this will have gradients proportional to

$$- nf\mathrm{H} \sin (nt + a) \qquad (14.12b)$$

since it is clear that when the cosine is at its maximum or minimum then the sine curve is zero, and when the cosine curve is changing most rapidly then the sine curve is a maximum or a minimum. The gradient curve gives a positive value when the cosine curve is rising. The factor n with fH is necessary because the rate of change in time of a cosine diurnal curve is only half that of a semidiurnal curve of equal amplitude.

We can express (14.12b) in the form

$$nf\mathrm{H} \cos (nt + a + 90°) \qquad (14.12c)$$

so that it can be placed on the machine by adding 90° to the original angles, provided that the original amplitudes are multiplied by factors proportional to the speeds. As we are interested only in the occasions when the gradient curve is zero, that is, the times of high water and low water, the values of nfH can be multiplied by any

arbitrary constant so long as they can be set to a suitable scale on the machine components.

The above gradient curve will be descending through zero (*i.e.*, the pen of the machine will be falling) when the gradient is changing from positive to negative, which corresponds to high water.

By a convention it is found simpler to make a rule that when the pen is descending through zero the time corresponds to low water, and when it is rising the time corresponds to that of high water, which requires us to set

$$nf\mathrm{H}\cos(nt + a - 90°) \qquad . \qquad . \qquad . \qquad . \quad (14.12\mathrm{d})$$

instead of (14.12c).

The two cards already prepared are (1) the permanent card of H, g (permanent, that is, until better constants are available) and (2) the card of values of $\mathrm{V} + u$ and f, which is available for all places for the year to which it pertains. A card is now prepared combining (1) and (2) so as to give the exact machine settings $\mathrm{R} = f\mathrm{H}$, $a = \mathrm{V} + u - g$, and $\mathrm{R}' = nf\mathrm{H} \times$ (the arbitrary constant). All these cards are very carefully checked, and compared with values obtained in the preceding year.

The machine is now cleared for action, and the zeros carefully set, both in amplitude and on the time scale. Then the amplitudes R′ and angles ($a - 90°$) are set, taking care not to turn the machine in doing so. All operations are independently checked, and the machine is now ready for use. The computer runs the machine until the height dial shows zero height, when the machine is stopped. This is effected by carefully designed methods of electrical switching so that the machine can be stopped at the exact point. The time is read from the time-scale, and it is noted whether the gradient curve is indicating high water or low water according to the convention mentioned above. Then the machine is re-started and stopped at the next zero, and so on.

After a few days of predictions have been obtained, a comparison is made with the over-run of the predictions for the preceding year. There should be exact agreement or very small discrepancies such as one minute, or two minutes at most in special cases. If this test is satisfactory, prediction proceeds until April 1, when the machine is stopped at zero hour and every setting is examined, and, if need be, adjusted for the variation in f and u. This test also confirms that all clutches have been properly tightened. Similar tests and adjustments are made on October 1 and January 1. The machine is run until January 6, so as to provide the over-run for comparison with subsequent predictions.

The machine is again brought to zero and then reset, with the normal precautions, with amplitudes R and angle settings a. It is then run to give heights. It is not necessary to stop the machine at the times of high and low water as in the case of the running for times, for the indications are easily read at the momentary halt at the reversal of motion, and there is ample time for the operator to enter the heights before the next indication is due. The same tests and adjustments are made on April 1, October 1, and January 1, and in addition at the beginning of each month there are tests on the times of high and low water. The machine is stopped at the same height just before and just after high water, and the average of the times is compared with the time obtained by the previous running of the machine. If the two agree within two minutes the test is regarded as satisfactory, otherwise it has to be reported and another test is made. This acts as a very stringent test on both runnings of the machine.

It will be seen therefore that very great precautions are taken to ensure strict accuracy, and it will be noted that though the machine is equipped to give graphical records these are not used in routine practice. The reasons are that curves are troublesome to prepare and to read. The time lines need to be drawn in for each zero hour by hand, then the reading of the charts to get heights and times is both tedious and inaccurate. It is almost impossible to ensure accuracy within five minutes in routine work when measuring times of high and low water from curves, for near the maxima and minima the pen movements are slow. The gradient, however, is changing most quickly at these times so that its accuracy is very much greater.

The 12 monthly tests made during the running of the machine for heights are always considered in relation to the inaccuracy inherent in the testing process, and the permissible discrepancies of only two minutes is regarded as a very stringent test upon *both* runnings. The tests, of course, are only taken with the higher high water of the day when there is pronounced diurnal inequality.

This double running of the machine, therefore, has very great advantages over a single running and a graphical record. The results obtained are ready for transcription on to official forms, after applying any shallow-water correction which may be found necessary.

The matter of shallow-water corrections is so important that a separate chapter will be devoted to it.

14.13. List of tide-predicting machines

The subject of tide-predicting machines has naturally been discussed principally in relation to the machines used for the Admiralty predictions, but the following list of machines may be of interest. It is arranged according to the date of construction, and includes all the machines known to have been made.

TABLE 14.1

List of Tide-predicting Machines

Date of completion	Designation	No. of components available
1873	British Association	10
1879	Indian	20—24
1881	French	15—16
1882	Ferrel (U.S.A.)	19
1908	Roberts	33—40
1910	U.S.A.	37
1910	Brazilian	12
1914	Japanese	15
1916	German	20
1918	Argentine	16
1924	Japanese	15
1924	Japanese	16
1924	Portuguese	16
1924	Tidal Institute	26
1927	Brazilian	16

In 1938 Germany completed a large machine (62 components), and between the years 1943–1948 machines of various types were constructed under the supervision of the Tidal Institute; for Russia (40), Norway (30), and Spain (16). Between 1948 and 1953 machines were constructed to a new design by A. T. Doodson and A. Légé & Co., for the Philippines (30), Thailand (30), Tidal Institute (42), India (42), and Argentine (42). The Tidal Institute's 1924 Kelvin machine (increased to 30 components), has been sold to France.

Three of these machines (1873, 1879, 1908) were made by A. Légé & Co., London, to designs by Sir William Thomson and Mr. E. Roberts in the first instance and by Roberts alone in the second and third. The third machine was purchased by the Liverpool Observatory and Tidal Institute in 1929 and reconditioned both then and later, and all the components are utilised. The French machine is so called because it was purchased by the French authorities in 1901; it was made by Kelvin and White, of Glasgow. The two U.S.A. machines and the German one were made for or by respective national authorities, and the rest have been constructed by Kelvin, Bottomley and Baird, of Glasgow. The 1914 Japanese machine was destroyed by an earthquake.

Many of these machines, of course, have been modified by the users for special purposes.

CHAPTER XV

SPECIAL PROBLEMS OF TIDAL PREDICTIONS

15.1. Tides in shallow water

TIDE-predicting machines are the most efficient means yet available for the prediction of tides, but in certain cases they fail to give predictions of adequate degree of accuracy. It has already been shown in Chapter VIII that the number of shallow-water constituents which demand consideration increases rapidly with the range of tide and that, for instance, a single constituent such as M_6 cannot adequately express the sixth-diurnal tide, and therefore when this tide is large it requires many constituents for which no machine yet built has made provision. Further, when the sixth-diurnal tide is large then the eighth-diurnal and tenth-diurnal tides and even higher species of tides cannot be ignored. The investigations made for Southampton in Chapter XXVI sufficiently indicate the complexity of the problems encountered.

One obvious solution would be to build a machine big enough to cope with all these problems, but the considerations of Art. 14.7 show that a large increase in the number of constituents would cause mechanical difficulties due to friction in the pulleys, though the problem is not an insoluble one in this respect, because all the additional constituents could be generated with amplitudes on a scale much greater than normal, and the sum of the results could be reduced mechanically in the appropriate ratio, which would also reduce the frictional error.

A more serious difficulty is that of "convergence," for unless the successive species of tide diminish fairly rapidly in range a few species will not be sufficient, and if it is necessary to include tenth, twelfth, fourteenth, and other species of tides there is little guarantee that higher species are not required.

While the shallow-water problem is mainly prominent in association with large semidiurnal tides (which happen to reach greater ranges than the diurnal tides) the problem is not unknown even with diurnal tides. In fact, third-diurnal tides are mainly generated in shallow water by interaction of diurnal and semidiurnal tides, as is shown in Chapter VIII, but in extreme instances it may be necessary to cope with fifth-diurnal tides and higher species.

15.2. General account of shallow-water corrections

A method has been devised, and is regularly used by the Tidal Institute for the solution of the problems described in the preceding article.

For simplicity, and in order to illustrate the general principles, suppose M_2 to be a predominant constituent and that the constituent S_2 is relatively small. We know that the effect of S_2 is made evident in springs and neaps, priming and lagging, that is, in a periodic variation of amplitude and phase-lag. Knowing the amplitude and phase-lags of the two constituents, it is a simple matter to express trigonometrically the variation in apparent amplitude and apparent phase-lag each day, and conversely if we tabulate these variations then we get an oscillation with a period of about 15 days, the exact period being that from one spring tide to another. The methods of analysis outlined in Chapter XIII will then express the perturbation of M_2 in the form :—

$$A \cos mT + B \sin mT$$

where T is the interval of time in units of a mean lunar day and m is the increment of phase in a lunar day. The analysis can be made, of course, for each of the two series of alternate high waters, or upon the two series of alternate low waters.

From any of these series, we could express the perturbation in the form

$$S \cos (W - c) \qquad (15.2a)$$

where S is the amplitude of the perturbation, W is the phase as indicated by the equilibrium constituents and c is the phase-lag. Obviously S and c are special harmonic constants.

Now consider the constituent MS_4. Its increment of phase in any interval of time, as is shown in Chapter VIII, is the sum of the increments of phase of M_2 and S_2 but in a mean lunar day the increment of phase of M_2 is 360°, which is ignored, so that the increment of phase of MS_4 in a lunar day is equal to that of S_2. Therefore, in any particular sequence of high waters or low waters, this compound constituent will yield perturbations of M_2, merging with and indistinguishable from the perturbations of S_2. Both high waters will yield the same effect, but the low water sequences will yield a different result, because the change of phase of MS_4 relatively to M_2 in six lunar hours is 180° greater than that of S_2. Hence if the perturbations of MS_4 and S_2 reinforce one another in the high water sequence, they will oppose one another in the low water sequence.

Again, consider the sixth-diurnal constituent $2MS_6$, whose phase increment in an interval of time, according to Chapter VIII, is equal to twice the increment of M_2, plus the phase increment of S_2. In a mean lunar day the phase increment of $2MS_6$ is equal to the phase increment of S_2, and again the two constituents will merge with one another in perturbing M_2. In this case, however, the perturbation due to $2MS_6$ will not be readily distinguishable from that due to S_2 because the phases increase by the same amount in six lunar hours. The only way of separating the two, if that were the object in view (which it is not) would be to work out the difference in the effects on the heights and times of high and low water. This would utilise a principle readily grasped from a consideration of the following formula.

Let

M_2 be denoted by $R \cos nt$,
S_2 ,, ,, ,, $A \cos nt + B \sin nt$,
$2MS_6$,, ,, ,, $A' \cos 3nt + B' \sin 3nt$,

no distinction being made between the speeds of S_2 and M_2 for the short interval of time considered. Then the value of $\cos 3nt$ is approximately the same as $\cos nt$ when t is small, while that of $\sin 3nt$ is equal to $3 \sin nt$. Hence the total tide is

$$(R + A + A') \cos nt + (B + 3B') \sin nt$$

approximately. If B and 3B′ are very small, the height of high water is given approximately by $(R + A + A')$ and the change in the time of high water is proportional to $(B + 3B')$. Therefore the sixth-diurnal constituent is three times more prominent, relatively to S_2, in the time variations than in the height variations.

Similar considerations apply to the constituents $3MS_2$, $3MS_8$, $3MS_4$ and a large number of other combinations of M_2 and S_2, and yet, however many the constituents are, they only behave like S_2 in perturbing M_2 except that the perturbations will not be the same in the high water sequences as in the low water sequences.

In effect, therefore, there are four sets of special harmonic constants of the type of (15.2a), provided that we are only dealing with shallow water as affecting semi-diurnal tides. These constants give perturbations in :—

(*a*) High water times;
(*b*) High water heights;
(*c*) Low water times;
(*d*) Low water heights.

If the diurnal tides are also prominent, we shall require eight such sequences because it will be necessary to differentiate between the two sequences of each of (*a*), (*b*), (*c*), (*d*).

In the analysis, therefore, for the perturbation discussed above, we treat the eight sequences separately and combine the results whenever possible. The methods

of analysis are very similar to those discussed in Chapter XIII, and in fact the multipliers used are exactly the same as are used in the monthly and annual processes of the general method of analysis of hourly observations.

15.3. Special harmonic shallow-water constituents

The next consideration which arises is, How many such perturbing constituents can be anticipated ? The answer is partially indicated by the lists of normal harmonic constituents and partially by the lists of shallow-water constituents, but it is needful to consider the cases where two constituents have equal and opposite values of phase increment in a lunar day. The value of m, the increment in phase per mean lunar day of 24·8412 mean solar hours, is given by

$$m = 24{\cdot}8412\, n - (0°, 360°, 720° \ldots) \qquad (15.3a)$$

where n is the speed in degrees per mean solar hour. Thus the constituents of L_2 and N_2 have values of m equal respectively to 13°·52 and −13°·52. It is necessary to alter the arguments of constituents with negative values of m so that

$$\cos(-13°{\cdot}52t - k)$$

say, is written as

$$\cos(13°{\cdot}52t + k)$$

Clearly, therefore, two constituents such as we have discussed will merge with one another.

The following table indicates the harmonic shallow-water constituents, which experience has shown to be worthy of consideration. It includes constituents derived from diurnal tides. The number of cycles per month in the perturbations is given in the second column, and the symbol for the constituent also indicates the approximate value of m, actually being the value of m after ignoring the decimals of a degree.

TABLE 15.1

List of Constituents of Harmonic Shallow-Water Corrections

Symbol	Cycles per month	m	W	Derivation	
C(00)	0	00°·000	—	$M_4, M_6, M_8 \ldots$	
C(01)	0	01°·020	h	Sa	
C(02)	0	02°·040	$2h$	Ssa	
C(11)	1	11°·713	$s - 2h + p$	ν_2, λ_2	and combinations with M_2
C(13)	1	13°·523	$s - p$	N_2, L_2	
C(25)	2	25°·236	$2s - 2h$	S_2	
C(27)	2	27°·276	$2s$	K_2	
C(36)	3	36°·949	$3s - 4h + p$	$2SN_6$	
C(38)	3	38°·759	$3s - 2h - p$	$2S\nu_6$	
C(50)	4	50°·472	$4s - 4h$	S_4	
C(52)	4	52°·513	$4s - 2h$	SK_4	
C′(11)	1	11°·598	$s - 2h$	P_1	and combinations with M_2
C′(12)	1	12°·618	$s - h$	S_1	
C′(13)	1	13°·638	s	K_1, O_1	
C′(27)	2	27°·161	$2s - p$	J_1, Q_1	
C′(36)	3	36°·834	$3s - 4h$	SP_3	
C′(38)	3	38°·874	$3s - 2h$	SK_3	
C′(40)	3	40°·915	$3s$	KO_3	

The values of m are the increments in phase per mean lunar day, and the values of W are given in terms of mean lunar longitude (s), mean solar longitude (h), and mean longitude of lunar perigee (p).

The values of the argument W are expressed in the same manner as the arguments for the normal tidal constituents (see Chapter VII), and for convenience we use the Greenwich values of W, so that there is a close analogy between the methods

used to express the constituents of the harmonic shallow-water corrections and the normal tidal constituents. The separate sequences of constituents for high or low water times or heights are all related to the same value of W which is therefore taken at the moment of mean lower lunar transit (*i.e.*, mean lunar midnight) at Greenwich.

Theoretically, each of the constituents varies with the revolution of the moon's nodes, but the determination of the appropriate corrections corresponding to f and u is not possible, owing to the many sources of derivation of the corrections. For example, C(00) would have a value of f compounded of the values of f for M_4, M_6, M_8 . . and the relative values of these constituents varies from place to place. In exceptional circumstances, it may be necessary to make some allowance for nodal variation.

15.4. Application to tides in shallow water

In the foregoing description of the harmonic shallow-water corrections it was supposed that constituent M_2 was predominant, and the question arises as to what the effects will be if, say, S_2 and other constituents are included in the primary prediction which is to be corrected. Theory proves that the differences between observed tides and primary predictions show the same periods as those indicated in Table 15.1, the reason for this being that as the whole of the arguments used depend essentially upon s, h, and p, any combinations of constituents only ring the changes on combinations of these three variables. The theory, of course, is very complicated, but it is confirmed by actual experience.

Hence, the normal procedure is to prepare primary tidal predictions using the predicting machine to its utmost extent, and then to compare the results with observations. The differences are analysed in sequences in relation to the lower transit of the mean moon, sequence 0 denoting the first high waters after the transit, sequence 1 the next low waters (being about six hours later than sequence 0), sequence 2 the next high waters (about 12 hours later than sequence 0), and sequence 3 the following low waters (about 18 hours later than sequence 0). For precision, sequence 0 is defined as being later than mean lunar transit by a time interval of hours of solar time given by

$$(g \text{ of } M_2)/29$$

It is clear that the shallow-water corrections so obtained depend upon the primary prediction, and supplement a particular set of harmonic tidal constants. Consequently, if later analyses are required, the same set of primary constituents must be used, for even if changes are taking place in the tides at the place considered, the changes are all allowed for in the corrections. Otherwise, all the previous analyses would be useless.

The normal method is not followed in cases where there are double high waters or double low waters, as the machine predictions might give double tides on some occasions and not on others. Under such circumstances, all normal shallow-water constituents are ignored in the primary prediction.

Further, we have supposed that the primary predictions are taken as for the place itself, but in estuaries it may be more convenient to take the primary predictions as for a place nearer the open ocean. Thus at Father point in the St. Lawrence estuary, shallow-water corrections are not required, as the machine predictions are quite satisfactory, but at Quebec the shallow-water effects are very pronounced. In such a case, seeing that the predictions for Father point are required in any case, it is possible to use these as the primary predictions and to analyse the differences between Quebec and Father point. This is advantageous for several reasons, financial and scientific. As a general principle it is better to use primary predictions which are simple by ordinary natural processes (such as in an oceanic type of tide) rather than an artificial simplification of an actual tide of estuary type.

15.5. Tides in long rivers

The shallow-water problems become acute in long rivers and a solution of the problems has been indicated at the end of the last article. As an example of the application of these methods we shall refer to the tides at Basra. This is 70 miles

up the Shatt-al-Arab, and the diurnal and semidiurnal tides are of comparable importance. The shallow-water effects are thus very complicated and the problem of prediction has been extremely difficult. The solution to the problem is to use as primary predictions the predictions computed annually for Shatt-al-Arab bar. These are of normal oceanic type with relatively small shallow-water effects, and when the differences between observations at the bar and at Basra were submitted to analysis for the constituents of Table 15.1, the resulting constituents provided corrections which have proved to be quite satisfactory. The method, of course, is an extension of that described in the preceding article for Quebec.

It is evident that these methods are the harmonic analogues of the non-harmonic method of differences, only the range of applicability is very much greater. The methods are exceedingly powerful and are used extensively by the Tidal Institute.

15.6. Use of predicting machines for corrections

The computation of the corrections is made possible by a special use of the predicting machine which we shall illustrate by reference to the constituent C(25), which has a phase increment per mean lunar day equal to that of S_2. If therefore we set the initial phase on the component for S_2 and run the machine, then successive readings at intervals of a lunar day will give the corrections for sequence 0, while the intermediate readings at intervals of half a lunar day will give corrections for sequence 2. But as half a mean lunar day corresponds to one revolution of the M_2 crank, we have an obvious method of computing the corrections: let all the shallow-water corrections be set on the appropriate components and take the readings whenever the M_2 crank is, say, pointing vertically downwards.

This method is also usable when the corrections C′ have to be applied.

The machine has thus to be set and run four times in order to produce the sequence of corrections for high water times, high water heights, low water times and low water heights, but the machine can be speeded up a little for this type of work.

Naturally, the cost of predictions is increased when corrections have to be made.

15.7. Other types of problems

There are many other types of problems which can be solved by the adaptation of methods described in this chapter. Occasionally approximate values of ordinary harmonic constants are desired from observations of high and low water, and a straightforward application of these methods is alone required. They can be applied also to yield predictions of the times of slack water of tidal streams, from observations of slack water, and the method is also applicable even if the observations have been taken only in daylight.

It is entirely beyond the scope of this Manual to give practical details of the methods as applied in specific problems, and such applications, of course, require great technical skill. Sufficient has been said to indicate the great variety of special problems which have been successfully attacked in recent years, and sufficient has also been said to explain the general principles utilised in these special methods.

One word of caution is desirable. Though it is possible to get values of M_2, S_2, and a few other principal constituents from high and low water, it is hopeless to expect to be able to develop a method which will give results of accuracy approaching that to be attained with less complications by the analysis of hourly observations. Any attempt to analyse high and low waters with thoroughness by these methods reveals a lack of "convergence." Thus the constituent S_2 in its effects upon high water height, for instance, will itself yield the constituent denoted by C(25) but will also yield constituents with values of m equal to integral multiples of that of C(25) and the amplitudes of these constituents will only decrease slowly. In fact we have in another form the same problem as we have in connection with the successively high species of normal shallow-water constituents such as M_4, M_6, M_8 . . . .

Thus, these special methods need to be applied with proper precautions, but when rightly used they greatly extend the applicability and power of the harmonic method of tidal predictions.

CHAPTER XVI

TIDE TABLES

16.1. Historical account of tide tables

PROBABLY the earliest recorded attempt at tidal prediction is that in the "Codex Cottonianus, Julius DVII," which exists in MS. in the British Museum. This work contains calendar and other astronomical or geographical information, some of which are the production of John Wallingford, who died Abbot of St. Albans in 1213, and, at p. 45b, a table on one leaf showing the time of high water at London bridge, "flod at london brigge," thus :—

Ætas Lunæ	h.	m.
1	3	48
2	4	36
3	5	24
..	..	..
..	..	..
30	3	0

In this table, column 1 gives the moon's age, columns 2 and 3 the time of high water corresponding to the age. As the time of high water when age is 0 or 30 days is the value of H.W.F. & C., which is now about two hours, it is evident that the time of high water at London has advanced about one hour since the thirteenth century.

In the "Philosophical Transactions," vol. xiii, Flamsteed published a table giving the time of high water at London bridge for the year 1683. As in the description of this table, he remarks "Hitherto our tide tables have only shown the time of one high water," etc., it is evident that tide tables for London were then regularly published.

Tide tables for the port of Liverpool, calculated by the Rev. George Holden, were first published in the year 1770, and have since been issued regularly.

It is probable that by this time tide tables were issued for several of the more important ports, but the methods used have not been divulged. Regarding this, Whewell, in the "History of the Inductive Sciences," says: "Liverpool, London and other places had their tide tables constructed by undivulged methods, which methods, in some instances at least, were handed down from father to son for several generations as a family possession, and the publication of new tables, accompanied by a statement as to the mode of calculation, was resented as an infringement of the rights of property."

Tide tables were first published by the Hydrographic Department of the Admiralty for the year 1833. The tables then consisted of a pamphlet of a few pages only, containing the predicted times of high water at the four principal ports in the British Isles. The development to the volumes now published has been continuous, as is shown by the following table giving the numbers of ports for which predictions have been provided, and the years when the most substantial changes were made.

Year	Home ports	Colonial and Foreign ports
1833	4	..
1836	10	1
1858	23	1
1910	26	6
1915	28	30
1929	29	90
1941	29	104

At first only high water predictions were given, but by degrees low water predictions were also included, until all "Standard" ports were provided with full predictions. The earlier volumes also included general information concerning tides and tidal streams, but in 1910 all tidal stream information was relegated to other publications, and in 1920 all tidal differences and non-harmonic quantities were issued in a separate volume (Part II) and revised at intervals of about five years. In 1927, harmonic constants were included in Part II, and in 1938 the whole of Part II was completely revised and data presented in the form required for the Admiralty method of prediction. An account of this method, along with other methods for prediction and analysis, was published separately in 1936 as Part III of the Tide Tables.

The rapid growth of the number of Standard ports, revealed in the above table, has now necessitated the issuance of Part I in two sections (A) and (B), the former for the British Isles and the north and west coasts of Europe, and the latter for the rest of the world. Further considerable developments are expected in the course of the next few years.

Other nations were not slow in following the example set by the Admiralty, and to publish official tide tables. Those of the French "Service Hydrograpbique" first appeared in 1839 and now give predictions for 51 ports, mostly in France and the French empire; those of the United States Coast and Geodetic Survey followed in 1853, and now give predictions for 106 ports in all parts of the world. Official tide tables are also published in India, Canada, New Zealand, Nigeria, and otber dominions and colonies, and in Japan, Germany, Brazil, the Argentine and other foreign countries.

16.2. Supplementary tables concerning the basis of predictions

The Admiralty Tide Tables contain a number of supplementary tables which give details of levels, non-harmonic quantities, authorities for the observations, constants and predictions, the method of prediction and the years of observation in which the predictions are based. Other miscellaneous information is also given relative to Land Survey datums.

This information is very interesting and valuable, and should be regarded as essential to the publication, for it enables an estimate to be made as to the relative value of the predictions. Generally speaking, while a year's observations suffice for predictions, there is added accuracy when the predictions are based upon several years of observation, and also when the observations are recent. But it does not follow that observations taken many years ago are not dependable, for much depends upon the character of the estuary in which the port is situated. Thus at Liverpool the analyses cover 11 years of observation, mainly about 1870, but including several analyses from recent observations. There is no perceptible change in the tide, as all the constants from recent observations agree remarkably well with those of 70 years ago. For London, the observations used are all in recent years, because it had been found that changes were taking place, probably due to the widening and deepening of the river channel.

In Section A the predictions are mainly based on observations taken in the present century, the exceptions, with the years of observations utilised, being given below :—

Devonport, 1882.	Swansea, 1858–61.
River Tees, 1897.	Galway, 1845–46.
Pembroke Dock, 1892.	

In Section B many of the predictions are based on very long series of observations, mainly because the harmonic method was enthusiastically developed for Indian ports. Thus we get, for example,

Karachi (1868–1920), 53 years.
Bombay (1878–1920), 43 years.

Also frequent analyses have been made for Canadian waters ; for example,

St. John, N.B. (1894–1917), 22 years.
Father point (1897–1921), 19 years.

In recent years the practice of analysing observations every year has been largely stopped. It is now considered necessary to analyse only at intervals of 3, 4 or 5 years, especially when many analyses have already been made for the port in question, and in India and Canada certain ports only are kept up to date in this way so as to keep a watch on any possible changes. It has also been realised that it may be more important to scrutinise the analyses (or comparisons of predictions with observations) to detect any periodic changes which might suggest the presence of constituents which have not been previously considered. It was found, for instance, that for certain places routine analyses were made year after year, without any improvement in prediction, whereas expert scrutiny revealed the necessity for additional constituents (such as shallow-water constituents) or for harmonic shallow-water corrections.

16.3. Supplementary tables to give heights at any times

The Tide Tables also give tables to assist in the calculation of the tide at hours intermediate between the times of high and low water. These tables are principally of value where there is very little diurnal inequality, for they depend upon the assumption of a harmonic type of oscillation. In other words, the tables give values of R cos nt for values of R at specified intervals. The proper value of n depends upon the period, and therefore upon the time of rise or fall of the tide (from low water to high water or *vice versa*). This time is measured from high water, and nt is denoted by θ.

Where there is mucb diurnal inequality, this method has a very limited value, and there is no simple method of getting intermediate heights with mixed tides (see Art. 16.4).

The harmonic assumption for the variation of height between high and low water also fails when there are large shallow-water tides, but fortunately, in such cases, as was shown in Chapter VIII, the shallow-water part is functionally related to the semidiurnal tide, and special tables can therefore be computed. The Admiralty Tide Tables give such special tables for places in the vicinity of the Isle of Wight.

Alternative forms of tables are given by certain port authorities, such as for Liverpool and London. These " reduction tables " are based upon many years of observations, and depend upon the assumption that the shape of the tide curve (if diurnal inequality is neglected) depends upon the range of tide. This assumption is fundamentally sound. The tide tables for Liverpool, published in Holden's Almanack, give reduction tables in terms of the interval of time, in units of half an hour from 6½ hours before the time of high water to 7 hours after it, and in terms also of the height of high water at intervals of a foot.

16.4. Tables of hourly heights

When the diurnal inequality is very great the only satisfactory method of giving the tide at any time is by the use of hourly heights, computed by the Admiralty method, or more elaborately by the tide-predicting machines. Such tables are prepared by the Tidal Institute for Penang, and tables for other places are prepared by foreign authorities.

For a number of years, tables of hourly heights were prepared by the Tidal Institute for Liverpool, and published in Holden's Almanack, but are now discontinued. It was found that it was very difficult to give adequate corrections for shallow-water tides, and the application of the harmonic shallow-water corrections to high and low water predictions at Liverpool was out of the question when the hourly heights could not be amended to conform to the improved high and low water predictions.

It was decided, therefore, that it was best to give the predictions of high and

low water to the greatest possible accuracy and to improve the reduction tables as much as possible. A further advantage of this is that different sets of reduction tables can be prepared for different places in the estuary. Thus, one set is available for the Landing stage, and another for Liverpool bay (for use at the bar), both being referred to the predictions for Liverpool. This flexibility has much to commend it.

16.5. Standard predictions and the system of interchange

It has been found highly desirable to have a system of interchange of predictions between the principal publishing authorities, and this not only on account of financial savings in the costs of prediction. One important reason is that there should not be two predictions existing for the same port, if it can be avoided, for predictions which differ from one another cause uncertainty and confusion, especially if there is no obvious guide as to which has the greater authority. Important legal cases touching matters of responsibility for loss of ships or cargo are often affected by arguments as to the use of the predictions. It is therefore desirable, for this reason and many others, that the best possible predictions should be made by the parties principally concerned, and made available to others who are also interested. Supplies from abroad are, however, liable to interruption, so the Hydrographic Department and the Tidal Institute are always prepared to compute all predictions at short notice.

This principle receives a very great application in the Admiralty Tide Tables. Thus, of the 133 ports included in the 1941 edition of the Tables, 20 are predicted in the Hydrographic Department of the Admiralty, and a further 17 at the Liverpool Observatory and Tidal Institute at Admiralty cost. The expense of predicting 14 ports at the Tidal Institute is shared by the Admiralty with various British and Colonial authorities, and 29 ports, 28 of which are predicted at the Institute, are supplied free of charge by British, Dominion, Colonial or Foreign authorities. The remaining 53 predictions are obtained from Indian and Foreign authorities in exchange for Admiralty predictions.

In addition to the Admiralty Tide Tables, there are many other unofficial tables published in this country, and until recently these tables were not identical with the Admiralty Tables. The methods used are mainly non-harmonic methods of a type no longer used for first-class predictions. Cheapness of production has influenced the publishers in accepting such predictions, but in recent years this has not needed to be a factor because the Hydrographic Department and the Stationery Office have made arrangements whereby copies of the Admiralty predictions can be supplied at very cheap rates so long as proper acknowledgment is made. It may also be stated that this enlightened policy militates somewhat against the sale of the official tables but is beneficial to a maritime country.

16.6. The accuracy of tide tables

It is a matter of great interest to enquire into the accuracy of tide tables. A general answer is not exactly given, for much depends upon the meteorological disturbances, and in some months the actual errors are quite small. As a general guide to the distribution of errors throughout the year in British waters we can take the frequency tables for Liverpool. In Table 16.1 we give the percentage of frequencies of errors in the times of high water and low water for each month of the year and in Table 16.2 we give the corresponding table of percentage frequencies in errors in heights.

In June there were in the high water times no errors greater than 10 minutes and in high water heights there were five months in which the greatest error was a foot. The predictions for May were remarkably good as regards the heights of high water, only 3% being greater than 0·5 ft. In February there were many stormy days, principally affecting the heights of both high and low water.

It will be noticed that 94% of the high water predictions and 85% of low water predictions are correct within 10 minutes, and that 90% of the high water predictions and 91% of the low water predictions are correct within 1 foot.

TABLE 16.1

Percentage Frequency of Errors in Time. Liverpool, 1937

Range of error in minutes.	High water times. 0 to 5	6 to 10	11 to 15	over 15	Low water times. 0 to 5	6 to 10	11 to 15	over 15
January	75	11	6	8	53	25	13	9
February	74	18	6	2	52	31	11	6
March	70	23	5	2	67	20	10	3
April	77	19	2	2	62	23	12	3
May	88	10	2	..	50	37	10	3
June	85	15	..	..	72	24	2	2
July	80	18	2	..	64	28	8	..
August	75	18	5	2	54	34	10	2
September	72	23	5	..	55	26	17	2
October	81	17	2	..	46	35	12	7
November	60	30	8	2	48	38	10	4
December	63	28	7	2	53	26	19	2
Average	75	19	4	2	56	29	11	4

TABLE 16.2

Percentage Frequency of Errors in Height. Liverpool, 1937

Range of error in feet.	High water heights. 0·0 to 0·5	0·6 to 1·0	1·1 to 1·5	over 1·5	Low water heights. 0·0 to 0·5	0·6 to 1·0	1·1 to 1·5	over 1·5
January	57	23	7	13	46	36	7	11
February	27	28	27	18	41	13	22	24
March	65	30	3	2	56	42	2	..
April	81	19	..	..	67	24	9	..
May	97	3	..	..	81	19	..	..
June	79	21	..	..	81	19	..	..
July	90	10	..	..	69	26	5	..
August	67	31	2	..	49	41	10	..
September	79	21	..	..	79	16	5	..
October	45	42	11	2	57	35	8	..
November	64	29	5	2	79	19	2	..
December	47	26	25	2	47	43	8	2
Average	66	24	7	3	63	28	6	3

It will be noticed that the average percentages for the greater errors in the heights are largely dominated by the errors during February. When considering these figures it is necessary to remember that the Irish sea is subject to large meteorological disturbances, and the year 1937 is not in any way exceptional in these respects in one way or another. Very great errors of over 5 ft. in the height of tide occur once in several years, while errors of 2 or 3 ft. may occur several times a year, the incidence being somewhat variable (see Chapter XXVIII, Surges).

Table 16.3 exhibits a table of percentage frequency of errors for Liverpool, Avonmouth (Port of Bristol, King road) and London. In comparing errors of predictions of heights regard must be paid to the range of tide. Thus at Avonmouth, situated near the head of an estuary, the convergence in breadth and depth which leads to a magnification of range of tide also leads to a magnification of meteorological disturbances. The only fair comparison between tables of errors for different

places is to consider percentages of errors for the same ratio in height. When this is done, the Avonmouth figures for errors in height become comparable with those for Liverpool. On the whole, however, it is evident from the errors in the times of high water and low water that the accuracy of the predictions is greatest for Liverpool, and that the predictions for London are not quite up to the standard for Avonmouth. Since 1934, however, further analyses have been made both for Avonmouth and for London.

TABLE 16.3

Percentage Frequency of Errors of Prediction in Liverpool, Avonmouth and London

Error.	Liverpool, 1937.		Avonmouth, 1934.		London, 1934.	
	HW.	LW.	HW.	LW.	HW.	LW.
Less than 10 minutes .	94	85	82	80	71	65
Less than 20 minutes .	98	97	96	95	96	88
Less than 0·5 ft. . .	66	63	49	37	60	61
Less than 1·0 ft. . .	90	91	81	71	87	77
Mean range . .	21 ft.		31 ft.		19 ft.	

The predictions considered in Table 16.3 all include harmonic shallow-water corrections. Without these the errors would be much greater. Thus, for the low water times at Liverpool (see Table 16.1), instead of the percentages

56, 29, 11, 4

we should have had

38, 33, 16, 13

Clearly the shallow-water corrections have greatly improved the low water predictions.

The predictions for other places in waters less liable to meteorological disturbances should not be less accurate than given above, though much depends upon the methods of analysis and prediction. It is necessary to point out, however, that where there is great diurnal inequality there may be only a few inches of height between adjacent high and low waters, and it is impossible to get accurate values of the times of these high and low waters. The only satisfactory method is to consider separately the higher high waters and the lower low waters.

It should be remarked that though, in this Article, differences between predictions and observations have been referred to as " errors," they are, in fact, mainly the effects of unpredictable meteorological disturbances.

CHAPTER XVII

PROGRESSIVE WAVES

17.1. The influence of terrestrial conditions

It has been shown that the variations in the forces can be expressed as sines or cosines, and this form of variation is the simplest natural form. The purest of musical notes can be shown to vary according to a sine or cosine law, whence the term " harmonic " is applied to this form of variation. It would be normally expected that if the force varies harmonically so also will the resulting tide, but experience shows, and theory confirms, that this is not strictly true. The variation in the tide is approximately but not truly harmonic, so that the response of the fluid to the force shows some measure of distortion.

Many analogies could be cited to illustrate how frequently distortion takes place in pbysical phenomena. It is well known that certain rooms have bad acoustic properties, and that the speech of a person whose utterance is " pure " and " clear " is heard imperfectly owing to the distortion caused by the shape of the room. The " pure " wireless waves are distorted by the inadequate response of the receiving set and loud-speaker. A disc whose natural period of oscillation is that of a tuning fork will not give the same quality of note, and a violin and a piano differ in quality because the notes they emit are not each pure harmonics but have distortional overtones.

Imperfection of response in the media is so general that it is not to be wondered at if the same susceptibility to distortion is found in the oscillations generated in the fluid medium with which we have to deal.

It will be necessary to discuss certain types of oscillations in some detail, and we shall commence with an account of the phenomena associated with the travel of a progressive wave. In later chapters we shall be concerned also with other types of oscillation, free and forced, progressive and stationary, but the theory of a free progressive wave offers for our purposes the simplest approach to the subject of distortion.

17.2. Streams associated with progressive waves

It is proposed to discuss the mode of propagation of a free wave along a channel of indefinite length in one direction only. Such a wave is called a progressive wave, and the term *free wave* is used to describe a wave which has been generated and which then moves independently of the generating force. The word *propagation* is used in the general sense of describing the mechanism of the movement, as distinct from the mode of generation ; in the case of a progressive wave the *rate of propagation* is the rate at which the wave travels. It is a matter of general knowledge that a disturbance of the water in a large pond or in a long channel will generate a wave which continues to travel after the original cause of the disturbance has ceased to operate, but there are many mistaken notions as to the mechanism of the movement, one being tbat a wave gives only a change in the elevation of the surface without any associated horizontal movement of the water itself. It is clear from observation that a wave does not carry with it in its onward course any small particles which may be floating on the surface, but it is the rapidity of travel of the wave which disguises the smaller rate of to and fro movement of the water itself.

Before proceeding to discuss the mechanism of propagation, we shall define certain expressions as follows :—

> *The length of a progressive wave* is the distance from crest to crest.
>
> *The period of a progressive wave* is the time in which the wave travels a distance of a wave-length ; or, at a fixed place, it is the interval of time between the occurrence of successive crests of the wave.

It is a simple matter to show that there must be a to and fro movement of the water. Referring to Fig. 17.1, suppose that a wave, whose profile is there represented, is travelling along a channel from left to right, and consider the volume of water between the sections of the channel at C and D, at each of which the wave surface is momentarily at the mean level. Between these two sections the wave is represented as everywhere below the mean level. Half a period later, the wave will have travelled forward so that the peak of the wave is then between C and D ; that is, the volume of water between the two sectional planes has increased considerably. This can only be effected by the actual flow of water inward into the space between the two planes. It also follows that at a later stage there must be an outward flow so that the flow of water is periodic with the wave itself.

The relation between the flow of water and the elevation of the surface is of very great importance, and it may be to some extent determined by considering the volume of water between the two sectional planes at points A and B in the channel, at opposite sides of the crest of the wave, and at equal distances from it. It should

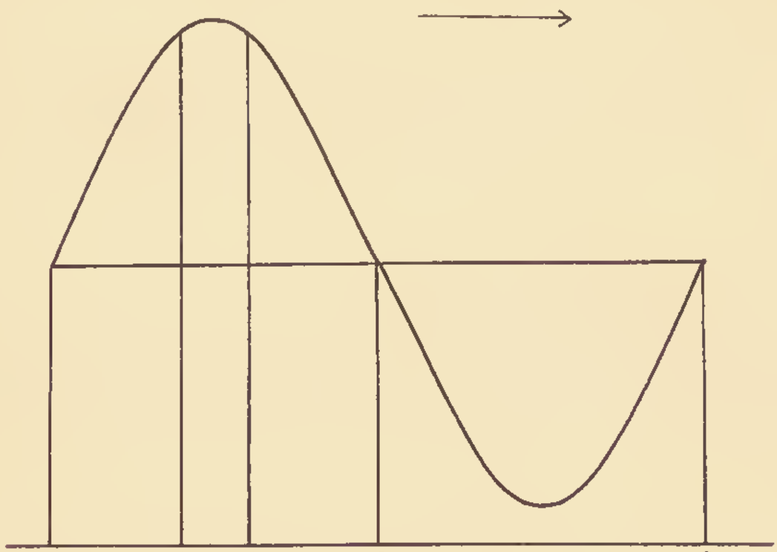

FIG. 17.1. Diagram for discussion of progressive wave.

be obvious that in this position the volume of water between the fixed planes A and B must then be a maximum, for it is high water between the planes. Under these circumstances, at this moment the total flux * of water across the section at A must be equally balanced by that across the section at B, for at the moment there is neither gain nor loss. But we have already seen that the motion of the water must be periodic with the wave, and it is easily demonstrated from a curve representing a periodic flow that the condition of equality of flux across A and B can only occur in three possible ways :—

(1) the flux at A and the flux at B are both zero ;
or (2) the flux half-way between A and B is a maximum, in the forward flow ;
or (3) the flux half-way between A and B is a maximum, in the backward flow.

The first possibility must be dismissed from further consideration because the two sections at A and B are typical sections and if the flux is zero for these it will be zero for all other chosen sections, and therefore the flux will be zero everywhere, but in such cases the tide cannot be rising or falling anywhere, which is not true.

* The word *flux* is used in connection with the transport of volume, no sign being associated with it, but the word *flow* is used of a stream whose direction may change in sign.

We have thus to decide between the second and third possibilities, but in either case we see that in a progressive wave the flux of water has its greatest value at high water of the wave, and we have to determine now the direction of the flow. An exactly similar argument for the minimum volume of water between two sectional planes on either side of the trough yields a similar result. Since, however, we have seen that the flow is periodic with the wave it follows that the flow at high water must be oppositely directed to the flow at low water.

Now consider the increase of volume between two other planes for some other phase of the wave, say between B and C, as the wave travels forward. This increase is at once accounted for on the supposition that the flow at the crest is in the forward direction and that at the trough is in the backward direction, so conspiring to produce the required increase of volume.

Hence we have deduced the following very important results characterising progressive waves :—

(1) the velocity of the water is periodic with the wave, so that at any particular place the water moves to and fro in the same period as the wave ;
(2) the flux of water is a maximum under the crest of the wave, and the flow is then in the same direction as the direction in which the wave is travelling ;
(3) the conditions are reversed at the trough, the flux of the water being then a maximum, and the flow is in the direction opposite to that of the travel of the wave.

17.3. The variation of the stream with the depth

In the foregoing discussion we have not assumed any law of variation of the rate of the stream with the depth, and we have been careful to deal only with the total flux of water across a section, wherever that flux may be taking place. It is now proposed to consider briefly the distinction between two classes of progressive waves.

In the first class, the length of the wave from crest to crest is small compared with the depth of the channel. Such are the conditions applicable to waves as experienced upon the surface of the sea, when the surface is disturbed with wind. It should be obvious that a disturbance of water over only a small area of the surface will tend to affect mostly the water in its immediate neighbourhood, and that in this class of wave, where the wave-length is short compared with the depth, the effects will become less apparent at great depths than they are near the surface, which is confirmed by mathematical theory.

In the second class of wave, the depth of the channel is small compared with the length of the wave from crest to crest. Such waves are called *long waves* and in this class are to be found all the tidal oscillations with which we shall be concerned. Under natural conditions, the length of a tide wave, from high water to high water is measured by hundreds of miles, whereas the depth is generally less than one mile, so that it is quite reasonable to consider tidal oscillations as coming under the class of long waves. Under such circumstances the extent of the surface affected by an oscillation is so great that the variation of the stream with the depth from the surface to the bottom of the sea can be considered as being negligible (apart from the effects of friction, which we shall ignore for the present). This is also confirmed by mathematical theory.

Hence the streams will be taken to be the same from top to bottom, and as we shall also consider the elevation to be small compared with the depth we can ignore any vertical components of velocity.

The conclusions arrived at concerning the relation of flux with elevation are thus also true for the relation between the stream and the elevation.

17.4. The rates of streams in progressive waves

It is shown later in this chapter by mathematical methods that the rate of the stream at any given point is approximately represented by

$$hu = cy, \quad \text{or} \quad \frac{u}{c} = \frac{y}{h} \qquad (17.4a)$$

where u is the rate of the stream ;
h is tbe mean depth of the channel ;
y is the elevation of tbe wave above the mean level ;
c is the mean rate of propagation of the wave.

It is possible by more elementary methods to show the truth of this relation, and it is desirable that an attempt be made to understand the pbysical concepts involved. In Fig. 17.2, take two sectional planes at A and C, where A and C are at a particular moment respectively at the crest and at the trough of the wave, XX′X″. Let B be the point in tbe channel half-way between A and C. Let the period of oscillation be denoted by T, so that in one quarter of the period the wave moves forward to the position shown by the wave ZZ′Z″. Suppose also that the channel has parallel vertical walls, and that it has a breadth denoted by b. Let tbe mean elevation from X to X′ above the mean level of the surface be denoted by Y, and let

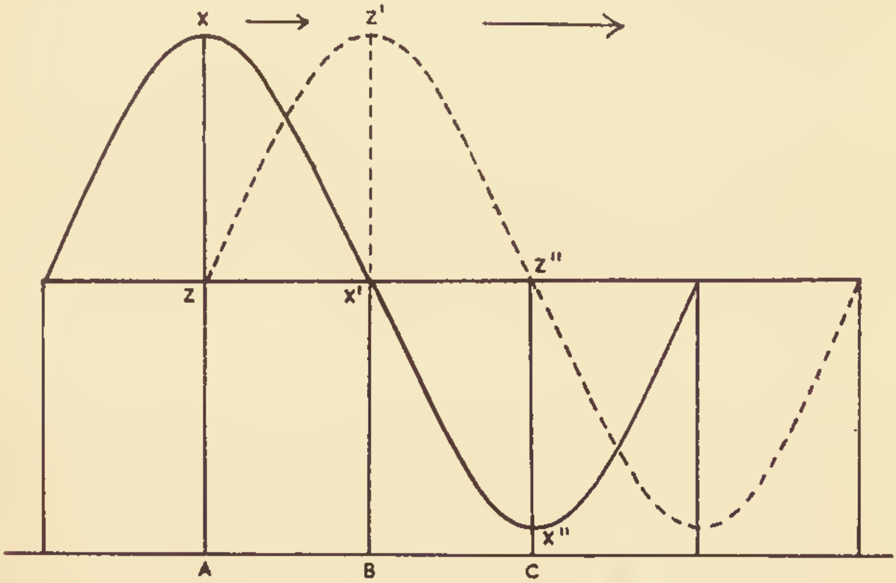

FIG. 17.2. Considerations of flux in progressive wave.

tbe mean velocity across tbe section at A in the quarter period be U. If the depth of the channel, denoted by h, be very much greater than the elevation of the wave above the mean level, then we can assume that the mean velocity across the section at the trough has the same magnitude and the opposite sign as the mean velocity across the section at the crest.

Now tbe flux across the section at A in the quarter pèriod $\frac{1}{4}$T is approximately

$$\tfrac{1}{4}\mathrm{T}(h + \mathrm{Y})b\mathrm{U} \qquad . \quad . \quad . \quad . \quad . \quad (17.4\mathrm{b})$$

and that across the section at C has the value

$$\tfrac{1}{4}\mathrm{T}(h - \mathrm{Y})b\mathrm{U} \qquad . \quad . \quad . \quad . \quad . \quad (17.4\mathrm{c})$$

yielding

$$\tfrac{1}{2}\mathrm{T}hb\mathrm{U} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (17.4\mathrm{d})$$

for the total increase of volume between A and C in the quarter period.

It is evident that as the wave XX′X″ moves forward to ZZ′Z″ there is a gain of volume between A and C to the extent of

(the mean elevation) × (the breadth) × (the length ZZ″).

The length ZZ″ is balf a wave-length, and it is equal to

$$\tfrac{1}{2}\mathrm{T}c \qquad . \quad . \quad . \quad . \quad . \quad . \quad (17.4\mathrm{e})$$

so that the total gain of volume is

$$\tfrac{1}{4}TcbY \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.4f)$$

This must be equal to the volume derived from the streams in the quarter period. whence we get

$$\tfrac{1}{4}TcbY = \tfrac{1}{4}ThbU$$

or

$$\frac{U}{c} = \frac{Y}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.4g)$$

Since U and Y are mean values, and since both u and y can be considered as harmonic expressions, and since both have their maximum values at the same instant as we saw in the previous article, it follows that $\frac{u}{y}$ must be constant, and that the value of the constant will be the value of $\frac{U}{Y}$.

Hence we get, as in (17.4a)

$$\frac{u}{c} = \frac{y}{h}$$

In Art. 17.7 we shall obtain the value of the rate of propagation c, and it is convenient to anticipate the results of that article in order to estimate the rate of a stream. Taking the case where the depth of the channel is 100 fathoms, the rate of propagation of the wave is there shown to be 82 nautical miles per hour, and if we then take the wave as having an amplitude of 10 ft., we get the maximum rate of the stream (at high water or low water) as being

$$\frac{10}{600} \times 82 \text{ knots} = 1{\cdot}4 \text{ knots.}$$

17.5. A wave as a reservoir of energy

Part of the mechanism of the propagation of a wave has been seen in connection with the relations between the stream and the elevation, but this needs to be supplemented by an understanding of the distribution of energy in the wave. All motion is a manifestation of energy, and no wave could ever be generated without work being done, resulting in the impartation of energy to the system. A coiled spring in compression is a store of energy and when the spring is released the energy is made manifest by the movement of the spring. During the compressed state the spring is said to have **potential energy,** and clearly there is also potential energy accumulated when a spring is expanded. There are thus two positions in which no movement is taking place and in which all the energy is in storage as potential energy. When the spring is released from either position there is rapid movement, and during this phase there is a diminution of potential energy and an accumulation of **kinetic energy,** as it is called, due to the movement of the spring. As the spring passes through its normal position the potential energy is zero, and all the energy is in the kinetic form.

This illustration could easily be supplemented by many others, such as the case of a swing. In its still position the swing has neither potential nor kinetic energy, whereas during the process of swinging there is a transformation of energy from the potential form to the kinetic form. At the top of the arc the energy is all in the potential form, and at the bottom of the arc all the energy is in the kinetic form. In general, all vibration may be considered as a periodic transformation of energy from the potential to the kinetic form, and from the kinetic to the potential form. Thus all waves can be considered as to their energy content, and the study of the relations between the potential and the kinetic forms of energy will throw much light upon the mechanism of propagation of the wave.

A wave in a fluid is more complex than a spring or a swing, for these can be considered as entities, whereas it is not always simple to consider a wave system as a

whole, especially when it is pictured as extending to an infinite distance. It is usual, therefore, to consider the energy of a wave within the compass of a wave-length.

Now if energy is not being provided from an outside source it is according to commonsense reasoning to conclude that in a vibrating system the total energy content remains constant, a principle called the Law of Conservation of Energy. As we are dealing with free waves, which are moving independently of the external forces, this principle will be utilised.

17.6. Expressions for the energy of a progressive wave

Before attempting to consider the energy of a wave per wave-length it is necessary to consider the energy in a small element of the fluid. Suppose that a progressive wave is travelling along a uniform channel bounded by vertical walls, and that the channel has a breadth b and a mean depth h. Let the elevation of the wave be denoted by y, and let the velocity of the fluid across a section transverse to the channel be denoted by u.

Consider firstly a small particle of mass m; its weight will be equal to mg, where g is the coefficient of gravitational force. The work done in lifting such a particle through a height y is mgy according to the ordinary definition of work as the product of the weight and the distance through which it is lifted, and this is also equal to the potential energy of the particle. If it has a velocity v then the kinetic energy is $\frac{1}{2}mv^2$, which the general reader can either accept or which can be verified from the discussion given later in Art. 17.9.

Now consider a small column of fluid on a base of area a, and having an elevation y. Its volume will be ay, its mass is the product of the volume and the density, and its weight is the mass multiplied by the coefficient of the force of gravity. If we suppose the water as having density ρ, then the weight of the column of water is ρgay, and this weight has been lifted from the mean level to average height equal to half the height of the surface of the column. Hence the potential energy of the column is

$$\tfrac{1}{2}y\,(\rho gay) = \tfrac{1}{2}\,\rho gay^2 \quad . \quad . \quad . \quad . \quad (17.6a)$$

The total area from crest to crest of a wave, in a channel of breadth b, is the product of the wave-length l and the breadth b, that is, the total area is lb, and the total potential energy per wave-length can be obtained from the expression for the potential energy of the small column of water just considered, if we replace a by lb and take the mean value of y^2 throughout the wave-length. The same result follows by considering a large number of columns covering the area from crest to crest. Hence the potential energy per wave-length is

$$\tfrac{1}{2}\,\rho glb \times (\text{the mean value of } y^2) \quad . \quad . \quad . \quad (17.6b)$$

In order to obtain the kinetic energy per wave-length, take an " element " of fluid bounded by two near planes separated by a distance d; the volume between the planes is practically equal to the product of the area of either plane and the distance between them, the mass is the volume multiplied by the density, and if the water has a velocity u, the kinetic energy, which is $\frac{1}{2}$ × (the mass) × (the square of the velocity), is thus

$$\tfrac{1}{2}\,\rho b(h + y)du^2 \quad . \quad . \quad . \quad . \quad . \quad (17.6c)$$

If we suppose that y is very small compared with h, we can use the simpler expression

$$\tfrac{1}{2}\,\rho bhdu^2 \quad . \quad . \quad . \quad . \quad . \quad (17.6d)$$

If we require the kinetic energy between two planes separated by a distance which is not small we must take the mean value of u^2 so that the total kinetic energy per wave-length is

$$\tfrac{1}{2}\,\rho bhl \times (\text{the mean value of } u^2) \quad . \quad . \quad . \quad (17.6e)$$

A relation between the potential and kinetic energies can be readily obtained by using the equation

$$hu = cy \quad . \quad . \quad . \quad . \quad (\text{from 17.4a})$$

whence we deduce

$$h^2 \times (\text{the mean value of } u^2) = c^2 \times (\text{the mean value of } y^2)$$

We then obtain from (17.6b) and (17.6e) the relation

$$\frac{\text{the potential energy per wave-length}}{\text{the kinetic energy per wave-length}} = \frac{gh}{c^2} \quad . \quad . \quad . \quad (17.6\text{f})$$

We now enquire as to whether there is any other relation between the potential and kinetic energies. It will be remembered that it was stated that all vibration may be considered as a periodic transformation of energy from the potential to the kinetic form, and from the kinetic to the potential form. This principle implies that the potential energy must be large enough to satisfy all the claims made upon it for the production of kinetic energy, and *vice versa*, and that on taking the average values throughout the period of vibration the average value of the potential energy must equal the average value of the kinetic energy. This is an important corollary to the princlple of the Law of Conservation of Energy, for the latter applies to non-vibratory states as well as to wave motions, but the corollary applies only to vibratory motion.

In the case of a spring the proof of this corollary follows easily from the fact that at certain stages the energy is entirely in one form, being entirely in the potential form at the extremes of contraction or of expansion and entirely in the kinetic form when the spring is passing through its mean position. But with a progressive wave no such simple states are found, so that we have to resort to an artifice, by considering two waves travelling in opposite directions, having equal amplitudes and periods. We have seen that at the crest of a wave the stream is a maximum in the direction of travel of the wave, and that at the trough the stream has an equal velocity backwards. If we consider the two waves at the moment when their crests coincide, then the streams pertaining to the oppositely moving waves will be everywhere in opposite directions and they will annul one another. At this moment, therefore, the combined system has no kinetic energy, and all the energy in the system is in the potential form arising from the elevation of the combined wave. This elevation is twice that of the single wave, and since the potential energy depends upon the squares of the elevations it follows that the potential energy of the combined wave is four times that of the single wave. Since the total energy of the combined wave is twice that of a single wave it follows that the total energy of the single wave is twice its potential energy. Therefore the kinetic energy must be equal to the potential energy.

17.7. The mean rate of propagation of a long progressive wave

We are now in a position to determine one of the most important expressions in the theory of tides, the relation between the rate of propagation of a wave and the depth of the water.

Since the potential energy of a wave has been shown to be equal to the kinetic energy it follows (from 17.6f) that

$$c^2 = gh \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.7\text{a})$$

where c is the mean rate of propagation of a wave;
g is the coefficient of gravitational force;
h is the mean depth of the channel.

It is desirable to note that in the derivation of this formula it has been assumed that the amplitude of vertical oscillation is small compared with the depth, otherwise we should have obtained an expression for c which would have involved the elevation. The value given above must therefore be regarded as the mean value for all states of the wave.

It is worth reiterating that this formula comes effectively from the theorem of Conservation of Volume yielding the relation (see 17.4a).

$$hu = cy, \quad \text{or} \quad \frac{u}{c} = \frac{y}{h} \quad . \quad . \quad . \quad . \quad . \quad (17.7\text{b})$$

and the theorem of Conservation of Energy together with the theorem as to the equality of the mean potential energy and the mean kinetic energy, yielding

$$\tfrac{1}{2} g\rho \times (\text{the mean value of } y^2) = h\rho \times (\text{the mean value of } \tfrac{1}{2} u^2) \quad . \quad (17.7c)$$

The left side of this equation expresses the average potential energy per unit of area covered by the wave, and the right side expresses the average kinetic energy in a column of fluid of unit cross-sectional area.

It is not absolutely necessary to assume that the depth of the channel is constant, but the proof is not of an elementary nature. It can be shown that the rate of propagation responds to the depth in such a fashion that the parts of the wave, such as the crest, can be considered as travelling momentarily at the rate appropriate to the depth immediately beneath them, and thus there is progressive distortion of the wave as it travels.

It is also worthy of notice that the rate of propagation of the wave is independent of the period of the wave, but the wave-length, of course, depends on the period, being equal to cT, where T is the period. Two waves having nearly equal periods will thus have nearly equal wave-lengths, so that two waves due to M_2 and S_2, for instance, will retain their relative amplitudes and phases over a large fraction of the wave-length. So also will K_1 and O_1, but the semidiurnal and diurnal waves will more quickly change their relative phases, since the wave-length of the diurnal wave would be about twice that of the semidiurnal wave. This principle has been used freely in earlier chapters, particularly in connection with the Admiralty Method.

It is of value to compute the rates of propagation, or of travel, for a few representative depths. Taking a unit of length as a foot, and the unit of time as a second, then the rate of travel for a depth of 1000 ft., according to the formula (17.7a), is

$$\begin{aligned} & \sqrt{32{\cdot}2 \times 1000} \text{ ft. per second} \\ &= 179 \times 60 \times 60 \text{ ft. per hour} \\ &= 179 \times 3600/6080 \text{ nautical miles per hour} \\ &= 106 \text{ nautical miles per hour.} \end{aligned}$$

Similar calculations give the following table :—

TABLE 17.1

Mean Rate of Travel of Progressive Wave

Depth in fathoms.	Rate of travel of wave (in nautical miles per hour).
50	58·17
100	82·27
200	116·34
500	183·95
1000	260·15
2000	367·90

For a semidiurnal wave with a period of 12 hours, the length of the wave for a depth of 100 fathoms is 82·27 × 12 = 987 nautical miles, whereas a diurnal wave with a period of 24 hours will have a length of 1974 nautical miles for the same depth.

17.8. The distortion of a wave

The mean rate of propagation of a wave, given by

$$c = \sqrt{gh} \quad . \quad . \quad . \quad . \quad . \quad (17.8a)$$

is independent of the elevation, and has been obtained (see Art. 17.7) on the assumption that the elevation is very small compared with the depth. In very deep water all parts of the wave will travel at the same rate, and distortion will not take place.

The more accurate mathematical investigations, as given later in this chapter, show that the rate of propagation is partly dependent upon the elevation, and is given by

$$c = \sqrt{g(h + 3y)} \quad . \quad . \quad . \quad . \quad . \quad (17.8b)$$

It is of some interest to enquire as to how it comes about that the rate so varies with the elevation. If the investigations of Art. 17.4 are re-examined it will be seen that the depth h only appears in association with y, in the form $(h + y)$. It is thus the actual depth at any moment, rather than the mean depth, which is of importance in the investigation. It is reasonable to suppose, therefore, that the rate of travel of a wave will be to some extent dependent upon the actual depth rather than on the mean depth, but it is not easy to deduce the actual law of variation.

It may also be noted that it is the flux across a section which is strictly proportional to the elevation and that the flux is proportional to the product of the velocity of the stream and the actual depth from the surface to the bottom. If the elevation varies according to a simple cosine law, so also will the flux, but the law of variation of the rate of stream will not be quite so simple. A curve exhibiting the variation of the rate of stream will show distortion as compared with the curve for the elevation. But as there is an interchange of energy continually taking place it follows that the elevation will also show a departure from its original simple law of variation.

The value for c shows that the crests of the wave travel faster than the troughs, so that at a particular point in the channel high water appears to be accelerated, and low water to be retarded, with the consequence that the tide rises more quickly than it falls. This is generally characteristic of tides in estuaries, and the consequences are so important that Chapter VIII was devoted to the subject.

17.9. Investigations of fluids at rest and fluids in steady motion

Before proceeding to the special theory of fluid motions, it is desirable to consider very briefly a few principles which are best exemplified by the case of a particle moving freely under gravity. Suppose a particle is at a certain height above the surface of the earth, and is allowed to fall under the influence of gravity. If g, as usual, is the coefficient of gravitational force, and t is the time elapsed, then the velocity acquired by a particle after time t is gt, since the steady application of the force will increase the velocity. The distance fallen in time t is $\frac{1}{2}gt^2$, and this can readily be verified as follows:—

The velocity at times $t = 0$, T 2T, . . . will be equal to 0, gT, $2g$T, . . . In the first interval, the particle will have a mean velocity of $\frac{1}{2}g$T, in the second interval a mean velocity of $\frac{3}{2}g$T, and so on. The distance travelled in the first interval will be $\frac{1}{2}g\text{T}^2$, in the second interval $\frac{3}{2}g\text{T}^2$ and so on. Hence at times $t = 0$, T, 2T, . . . the distances travelled are respectively

$$\begin{array}{llllll} & 0, & 0{\cdot}5g\text{T}^2, & (0{\cdot}5 + 1{\cdot}5)g\text{T}^2, & (0{\cdot}5 + 1{\cdot}5 + 2{\cdot}5)g\text{T}^2 \ldots \\ \text{or} & 0, & 0{\cdot}5g\text{T}^2, & 2g\text{T}^2, & 4{\cdot}5g\text{T}^2 \ldots \end{array}$$

and these are values of $\frac{1}{2}gt^2$, for $t = 0$, T, 2T, . . .

A particle acquires energy according to the work done on it, and if a particle of mass m and therefore of weight mg is lifted through a distance y, then the work done on it is equal to

$$mgy \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.9a)$$

and this is also equal to the potential energy, because the particle can thence acquire kinetic energy by falling through the distance y. The potential energy is thus always defined with reference to some datum plane or position. The particle will fall a distance given by $\frac{1}{2}gt^2$ in the time t and will then acquire a velocity $v = gt$, as has already been stated. Hence the particle falls through tbe distance y in a time given by

$$y = \tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad . \quad . \quad (17.9b)$$

so that

$$mgy = \tfrac{1}{2}mg^2t^2 = \tfrac{1}{2}mv^2 \quad . \quad . \quad . \quad . \quad . \quad (17.9c)$$

Hence the potential energy of a particle at height level y is equal to the kinetic energy which the particle would acquire if it fell from rest through the distance y under the influence of the force of gravity.

The earth's gravitational force is simply a particular instance of a force, and in general every force with which we shall be concerned is said to have a potential.

The work done by a pressure of intensity p (the pressure per unit area), exercised over an area s, through a distance x, is similarly equal to

$$psx \qquad \text{(17.9d)}$$

In the case of a fluid at rest the pressure intensity experienced at any point of the fluid is simply given by

$$p = \rho h \qquad \text{(17.9e)}$$

where ρ = the density of the fluid;
h = the depth of the point below the free surface.

It is a consequence of the mobile properties of a fluid that the pressure at the point is the same in all directions, as may be seen from the fact that if a small tube is

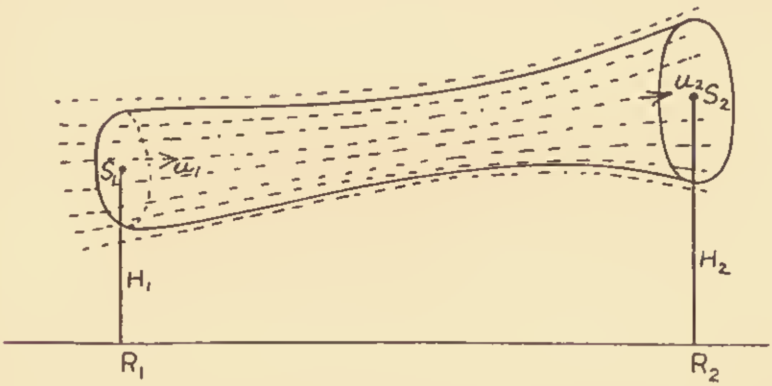

FIG. 17.3. Stream-lines and stream-tube.

inserted into the liquid then the liquid will rise to the same height in the tube (that is, the level of the surface of the fluid outside the tube) whatever be the direction of the end of the tube in the liquid.

Passing now to fluids in motion, the simplest state of fluid motion is that in which the motion is *steady*. The actual paths of the particles are called *stream-lines*, and these preserve their configuration unchanged. If a tube could be made so that it varied in its cross-section according to the shape of the stream-lines, then it could be inserted in the fluid without disturbing the motion (see Fig. 17.3). Suppose that this *stream-tube* has an area s_1 at R_1, taken transversely to the stream, and let the particles passing through the tube arrive at the point R_2, where the area of the cross-section is s_2, after a small interval of time t. Let the velocities at R_1 and R_2 he u_1 and u_2, respectively. Then the volume of water which enters the tube over the area s_1 in the time t will be

$$V = u_1 s_1 t \qquad \text{(17.9f)}$$

Since the same volume of fluid must traverse each cross-section in the same time, then

$$u_1 s_1 = u_2 s_2 \qquad \text{(17.9g)}$$

Now let the pressure intensity (that is, the pressure per unit area), at R_1 and R_2 be denoted by p_1 and p_2, respectively, and let the mean heights of the areas s_1 and s_2 be denoted by H_1 and H_2, respectively, relative to some fixed horizontal plane. Then the pressure over the area s_1 is $p_1 s_1$. This helps to move the volume of water V

through a distance u_1t, so that it does work equal to the product of the force by the distance through which it is exerted, and this is

$$(p_1s_1)(u_1t) \qquad (17.9h)$$

which is clearly equal to

$$p_1V \qquad (17.9i)$$

At the area s_2, however, the work done *against* the pressure p_2 is

$$(p_2s_2)(u_2t) = p_2V \qquad (17.9j)$$

Hence the total work done in the direction R_1R_2 is equal to

$$(p_1 - p_2)V \qquad (17.9k)$$

Now the volume of water entering at R_1 will have a kinetic energy of $\frac{1}{2}\rho V u^2{}_1$ and that leaving at R_2 will have a kinetic energy of

$$\frac{1}{2}\rho V u^2{}_2 \qquad (17.9l)$$

so that the gain of kinetic energy is

$$\frac{1}{2}\rho V(u^2{}_2 - u^2{}_1) \qquad (17.9m)$$

But the volume of water entering at R_1 and leaving at R_2 will gain potential energy equal to

$$\rho Vg(H_2 - H_1) \qquad (17.9n)$$

since ρV is the mass of the volume of fluid.

If there are external forces, such as tide-generating forces, these also will have a potential P, say, and the gain of potential energy from this source can be written as

$$\rho V(P_2 - P_1) \qquad (17.9p)$$

The work can now be equated to the increase of energy, so that

$$(p_1 - p_2)V = \frac{1}{2}\rho V(u^2{}_2 - u^2{}_1) + \rho Vg(H_2 - H_1) + \rho V(P_2 - P_1)$$

or

$$p_1 + \frac{1}{2}\rho u^2{}_1 + \rho gH_1 + \rho P_1 = p_2 + \frac{1}{2}\rho u^2{}_2 + \rho gH_2 + \rho P_2 \qquad (17.9q)$$

If a further element be taken from R_2 to R_3 a similar equation will be obtained connecting p, u, and H at the two ends of the new element of tube. Hence, along a stream-line

$$\frac{p}{\rho} + \frac{1}{2}u^2 + gH + P = \text{a constant} \qquad (17.9r)$$

The constant will vary with each stream-line.

This extremely valuable formula was given by Bernoulli in 1738, and it has many applications. It may be remarked that it is to some extent an extension of the Law of Conservation of Energy, in that it asserts that along a stream-line the sum of the pressure-intensity and the kinetic and potential energies per unit mass is a constant.

17.10. The application of Bernoulli's equation to progressive waves

At first sight a progressive wave is not a case of steady motion, but that is because we instinctively consider ourselves as resting on the earth while the wave travels past us. If we suppose ourselves to be travelling at a particular point of the wave we should consider the wave profile to be stationary, while the water would appear to be moving *backwards* relatively to the direction of travel of the wave, and the motion would be *steady* in the absolute sense if all parts of the wave travelled at the same rate so that the profile retained its shape indefinitely. We have reason to suppose that the changes in the profile are comparatively slow, so that we are justified in supposing the motion to be steady over such an interval of time as will allow

the application of Bernoulli's equation. That equation was obtained on the supposition that within the limits of the tube and for a short time the motion was truly steady. The artifice of travelling with the wave and relating all velocities to the moving point should cause no real difficulty, because we always deal with relative movements, and not absolute movements. Any moving particle on the earth has an absolute velocity composed of its velocity relative to the earth, the velocity of the earth relative to the sun, and the sun's velocity in space, relative to the so-called fixed stars, and so on.

For simplicity, suppose that the wave is travelling in a long channel or canal of constant breadth b, and that the undisturbed surface of the water has a depth h, and suppose that the canal has vertical walls. Let the elevation of the surface of the water above the mean level be denoted by y, let the velocity of stream relative to the bed of the channel be u, and let a small portion of the wave profile be supposed to be travelling, relatively to the bed of the channel, at a rate denoted by c.

In Fig. 17.4, let two sections transverse to the channel be taken at two points R_1 and R_2 on opposite sides of the point R, all three points being taken in the profile

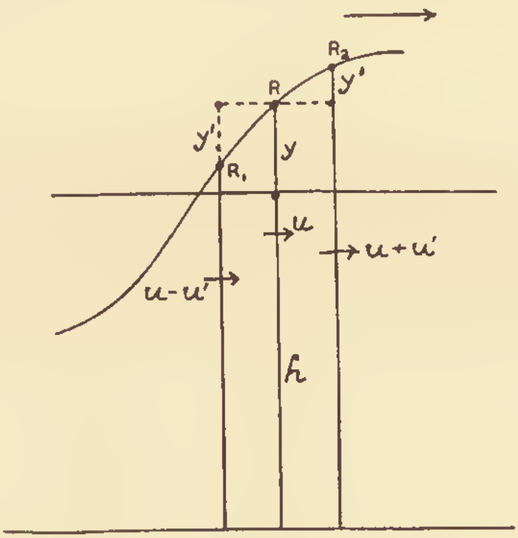

FIG. 17.4. Flux across sections of channel.

of the wave, which is supposed to be travelling from left to right. Let R be at equal distances from R_1 and R_2, let the elevation at R be denoted by y and let R_1 and R_2 be so close to R that the elevations at R_1 and R_2 may be denoted respectively by $y - y'$ and $y + y'$, where y' is very small. Also, let the velocities of the stream, relative to the bed of the channel, be denoted by $u - u'$, u and $u + u'$ at the three points. We shall suppose that since the three points are very close together that they are effectively travelling at the same rate c.

If we now suppose ourselves to be travelling with the wave, then the apparent rates of stream, relative to the profile, become respectively,

$$u - u' - c, \quad u - c, \quad \text{and} \quad u + u' - c.$$

Since the wave surface is supposed to be fixed and the water to be flowing past it, then the profile of the wave is a stream-line, to which we can apply Bernoulli's Equation, with the simplified condition that the pressure p is zero, if we ignore the atmospheric pressure. There are no external forces to be considered so that P is also zero, whence we simply get

$$(u - u' - c)^2 + 2g(y - y') = (u + u' - c)^2 + 2g(y + y')$$

which gives

$$(u - u' - c)^2 - (u + u' - c)^2 = 4gy'$$

or

$$(u - c)u' = - gy' \quad . \quad . \quad . \quad . \quad (17.10a)$$

There is another condition to be satisfied in connection with the volume of water passing the three sections, for this must be the same for each.
Hence we get

$$b(h+y-y')\,(u-u'-c) = b(h+y)\,(u-c) = b(h+y+y')\,(u+u'-c) \quad . \quad . \quad (17.10b)$$

The first and third of these, after equating and multiplying out, give

$$(h+y)\,(u-c)-y'(u-c)-u'(h+y)+u'y'=(h+y)\,(u-c)+y'(u-c)+u'(h+y)+u'y'$$

and therefore

$$(h + y)u' = - y'(u - c) \quad . \quad . \quad . \quad . \quad (17.10c)$$

On substituting by this equation into (17.10a) we get

$$(u - c)^2 = g(h+y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.10d)$$

whence

$$c = u + \sqrt{g(h + y)} \quad . \quad . \quad . \quad . \quad . \quad (17.10e)$$

or

$$c = u + \sqrt{gh}\sqrt{1 + \frac{y}{h}} = u + \left(1 + \frac{y}{2h}\right)\sqrt{gh} \quad . \quad (17.10f)$$

if $\frac{y}{h}$ is small.

In this expression the most important part is given by $\sqrt{gh}$, which we usually denote by c, and we have had evidence to show that $\frac{u}{c}$ and $\frac{y}{h}$ can be regarded as small quantities, and that u is approximately proportional to y. In order to eliminate u from the last equation it is necessary to determine the relation between u and y, which is done by using equation (17.10b) and noting that the average volume passing all sections is obtained by taking the average relative velocity as $-c$, the average elevation as h, so that we get the rate of transport of water across the sections must equal $-bhc$.

Hence

$$b(h + y)\,(u - c) = - bhc$$

and therefore

$$u - c = - \frac{hc}{h + y}$$

whence we have the relation between the elevation and stream in the form

$$u = \frac{cy}{h + y} \quad . \quad . \quad . \quad . \quad . \quad (17.10g)$$

To a first approximation, therefore,

$$\frac{u}{c} = \frac{y}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.10h)$$

This result justifies all the approximations made, for if $\frac{y}{h}$ is small, so also is $\frac{u}{c}$.

Substituting in (17.10f) we get

$$c = \left(1 + \frac{3y}{2h}\right)\sqrt{gh} \quad . \quad . \quad . \quad . \quad (17.10i)$$

This gives the rate of propagation of the wave at different elevations.

An alternative method of obtaining this relationship gives

$$c = \sqrt{g(h + 3y)} \quad . \quad . \quad . \quad . \quad (17.10j)$$

and it is obvious that (17.10i) is equivalent to this if $\frac{y}{h}$ is small.

The consequences of this variation in the rate of propagation of a wave, for different parts of the wave, were discussed in Art. 17.8.

CHAPTER XVIII

STANDING OSCILLATIONS AND REFLECTED WAVES

18.1. Standing oscillations

THE theory of a progressive wave has been necessarily developed on the supposition that the wave never encounters any barrier, and therefore the wave must either travel in an infinitely long channel or else in a re-entrant channel such as a canal encircling the earth. The existence of land masses on the earth as we know it forces us to attach great importance to their influence. While it is possible to consider the effects of an obstruction upon a progressive wave, and thence by stages to approach the subject of the oscillations which can occur in a basin which is entirely landlocked, it is preferable that the oscillations in such a basin shall be considered in a more direct manner. After this has been done the relations between progressive waves and standing waves will be considered.

It is a matter of observation that if a dish containing water is disturbed then the water will oscillate in a see-saw fashion. The vertical movements will be greatest at the extreme edges of the dish, and if the dish has a simple shape there will be a line near the centre of the dish, corresponding to the pivot of the see-saw, over which no vertical movement will take place. Such a line is called a *nodal line*.

Consider the sequence of events, commencing at the moment when the surface is everywhere momentarily horizontal, and is about to rise on the right of the nodal

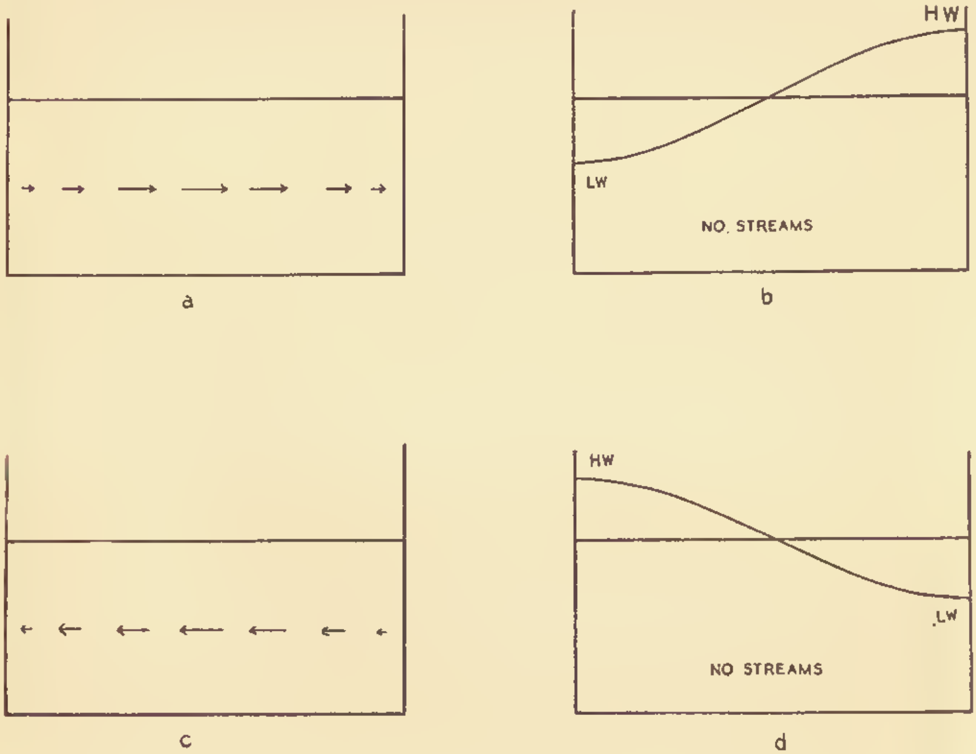

FIG. 18.1. Tides and streams in simple standing oscillation.

line. It is obvious that there must be a stream flowing across the nodal section (the section of the basin under the nodal line) in order to increase the volume in the right half of the basin, and that as the elevation rises there towards high water the volume of water requires to be transported at a decreasing rate as the rate of rise slows down. After high water the volume of water in the right half of the basin must begin to diminish, so that the stream across the nodal section must turn at high water. If we considered any other section we should get a like result, and it would be clearly seen that high water occurs at the same moment everywhere on one side of the nodal line. Hence a standing oscillation is characterised by the occurrence of slack water at high and low water, whereas with a progressive wave the stream is strongest at the moments of high water and low water. It follows that if the oscillations are of the simple type we have been considering, the greatest rates of stream will occur half-way between the times of slack water, and therefore with a standing oscillation the streams are strongest at half-tide.

These conditions are illustrated in Fig. 18.1, commencing with the half-tide condition prior to the tide rising on the right. The surface is level, and the streams are at their strongest everywhere, though the rates vary according to the distances from the ends. The greatest rate occurs over the nodal section and the streams are weakest near the bounding ends, across which no flow can take place. This variation of rate is shown graphically. After a time equal to one-quarter of the period there will be a state of high water to the right of the nodal line, and low water to the left, and there will be no streams anywhere. The diagrams for the next two quarter-periods are the exact reverse of the first two diagrams.

18.2. The period of a standing oscillation

For simplicity, suppose that the basin has a rectangular shape, with length L breadth b, and mean (or undisturbed) depth h. Let the elevation above the mean surface be denoted by y, let the rate of the stream be denoted by u, and let the period of oscillation be T.

At the moment of high water, the volume of water raised above the mean level, say to the right of the nodal line, is obviously given by

$$\tfrac{1}{2}b\mathrm{L} \times \text{(the mean value of } y \text{ at high water)} \qquad (18.2a)$$

where

$\tfrac{1}{2}b\mathrm{L}$ is the area of half the basin.

Now this volume of water has been transported across the nodal section in the quarter-period from half-tide to high water, and the volume of transported water is clearly equal to

$$\tfrac{1}{4}\mathrm{T}bh \times \text{(the mean value of } u_n \text{ in the quarter-period)} \qquad (18.2b)$$

where bh is the area of the nodal section, and u_n is the stream at the node.

Since y increases from zero at the node to a maximum value, Y say, at the extremity of the basin, and since u_n decreases from a maximum value, U say, to a zero value in the quarter-period, it is reasonable to suppose that the two means are similarly related to the two maximum values, so that we shall feel justified in writing

$$\frac{\text{the mean value of } y \text{ at high water}}{\text{the mean value of } u_n \text{ in the quarter-period}} = \frac{\mathrm{Y}}{\mathrm{U}} \qquad (18.2c)$$

Then, since the volume of water given by (18.2a) must be equal to that given by (18.2b), we get

$$h\mathrm{UT} = 2\mathrm{LY} \qquad (18.2d)$$

Now consider the energy relations. At high water there are no streams, so that all the energy is in the potential form, and it is readily seen, as in Art. 17.6, that this is equal to

$$\tfrac{1}{2}gb\mathrm{L}\rho \times \text{(the mean value of } y^2 \text{ at high water)} \qquad (18.2e)$$

At half-tide, all the energy is in the kinetic form, which is given by

$$\tfrac{1}{2}hbL\rho \times \text{(the mean value of } u^2 \text{ at half-tide)} \quad . \quad . \quad . \quad (18.2f)$$

It needs to be noted that both the means in the last two equations are taken over the whole basin. Now in (18.2e) the value of y changes from $-$ Y at the left end, through zero at the node, to the maximum value Y at the right end of the basin. Similarly, in (18.2f) u ranges from zero at the left end, through the maximum value U at the node, to zero at the right end. Each half of the basin contributes half the energy, and it is simpler to consider the variations of y and u in one-half of the basin. We have seen that y^2 increases from zero to Y^2, and u^2 decreases from U^2 to zero. It is reasonable to suppose that

$$\frac{\text{the mean value of } y^2 \text{ at high water}}{\text{the mean value of } u^2 \text{ at half-tide}} = \frac{Y^2}{U^2} \quad . \quad . \quad (18.2g)$$

On applying the Law of Conservation of Energy the expressions (18.2e) and (18.2f) are equal, and on using (18.2g) we get

$$gY^2 = hU^2 \quad . \quad . \quad . \quad . \quad . \quad (18.2h)$$

But (18.2d) gives

$$4L^2Y^2 = h^2U^2T^2 \ . \ . \ .$$

so that we get

$$4L^2 = ghT^2$$

or

$$T = \frac{2L}{\sqrt{gh}} \quad . \quad . \quad . \quad . \quad . \quad (18.2i)$$

This formula gives the period of oscillation in terms of the length of the basin, L, and the wave-rate $\sqrt{gh}$. The method of deduction is similar to that used in discussing a progressive wave, for which we obtained the period as $l/\sqrt{gh}$, where l is the length of the wave. It will be noticed that the formula for the standing oscillation shows that the period of oscillation is equal to that for a progressive wave whose length is equal to twice the length of the basin.

This is a very interesting fact and its import will be considered in the next article but one, after considering the reflection of a progressive wave.

18.3. Reflected waves

The similarity between the formulæ for the periods of progressive waves and standing oscillations shows that there is a definite relationship between the two types of movement. To elucidate this, suppose that we have two systems of progressive waves of equal amplitudes, moving in opposite directions in a canal of infinite length. Their rates of travel will, of course, be equal, and we shall suppose that they have the same wave-length. We shall ignore variations in the rate of propagation by supposing the elevation of the wave to be small compared with the depth.

In Fig. 18.2 we give the positions of two of these waves at various stages of their movements. The wave travelling to the right is indicated by the full line, and the wave travelling to the left is indicated by the broken line. In position (a) the two waves are coincident; in position (b) each has moved through $\frac{1}{8}$th of a wave-length; in position (c) the waves have moved through another $\frac{1}{8}$th of a wave-length, and positions (d) and (e) show further movements. The whole diagram therefore shows successive movements of the waves through a distance of half a wave-length.

Now it was shown in Art. 17.4 that the stream and elevation in a progressive wave are related by the formula

$$hu = cy$$

At the points on the vertical line in the diagram, denoted by BB, the elevation is the same for both waves, and since the value of u was measured in the direction of propagation it follows that at the points on BB the two waves will have equal and

opposite values of water velocity; that is, there is an annulling of water velocity at the points BB.

Since this is true for all states of the oscillation, it follows that there is never any movement of fluid across the plane at B transverse to the channel, so that a barrier could be placed at B without affecting the oscillatory movements on either

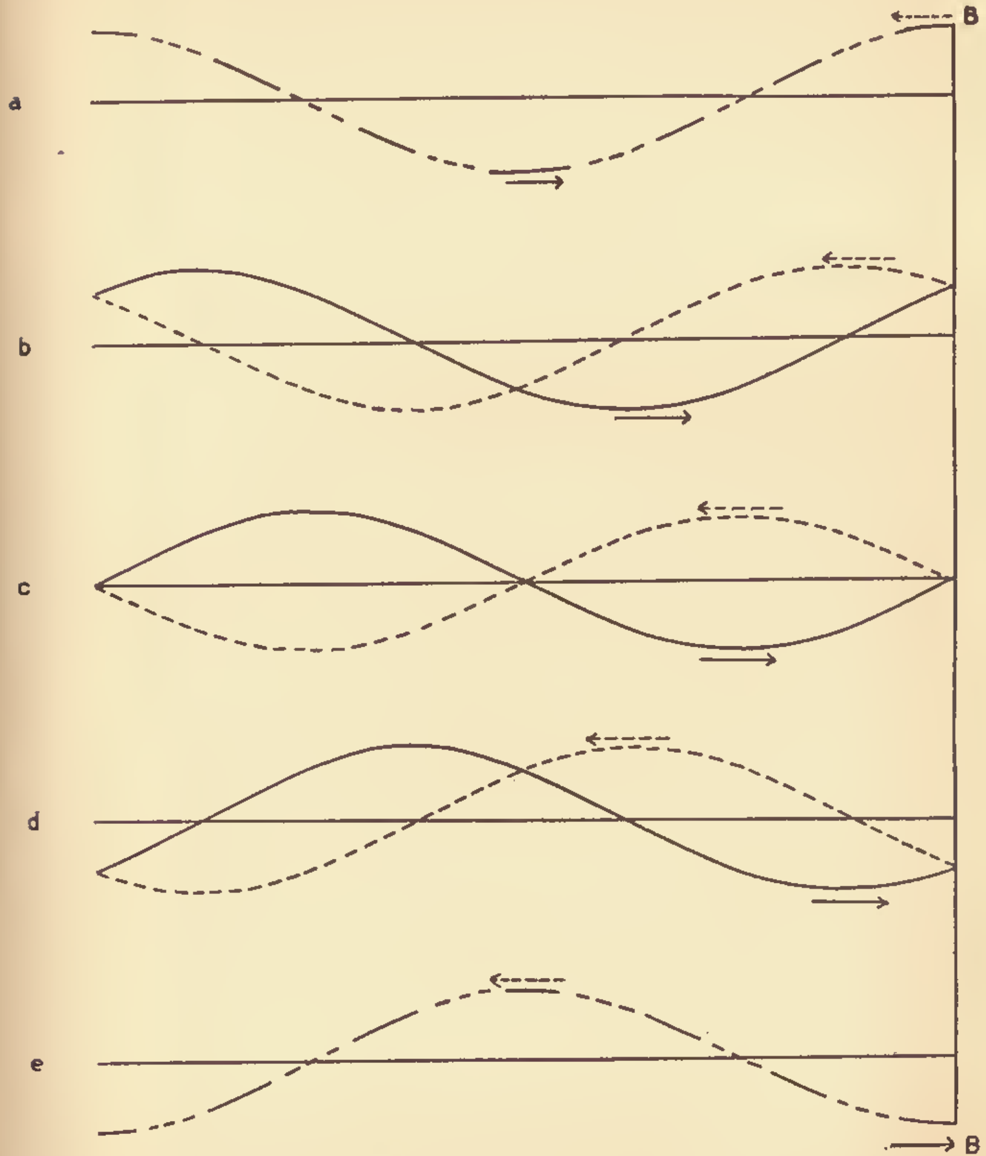

FIG. 18.2. Waves moving in opposite directions.

side of it. Hence we see that the wave travelling to the left, away from B, can be considered as due to reflection of the wave travelling to the right, towards B.

Some further deductions can be made from the formula connecting the stream and the elevation. The wave travelling to the right will be called the primary wave and that travelling to the left will be called the reflected wave. In position (a) the two waves have the same elevation at any point, and the streams are equal and opposite, and therefore annul one another, so that it is slack water everywhere. A

similar result is found in position (e). In position (c), since the primary wave gives a trough to the left of the barrier, the stream is flowing in the opposite direction to that of the wave, so that the stream is flowing to the left. But the stream due to the elevation in the reflected wave is also flowing to the left, so that though the elevations annul one another the streams reinforce one another. Further, at the centre of the trough of the primary wave the velocity is greatest, and it is clear that at this point the conditions are most favourable for reinforcement of velocities.

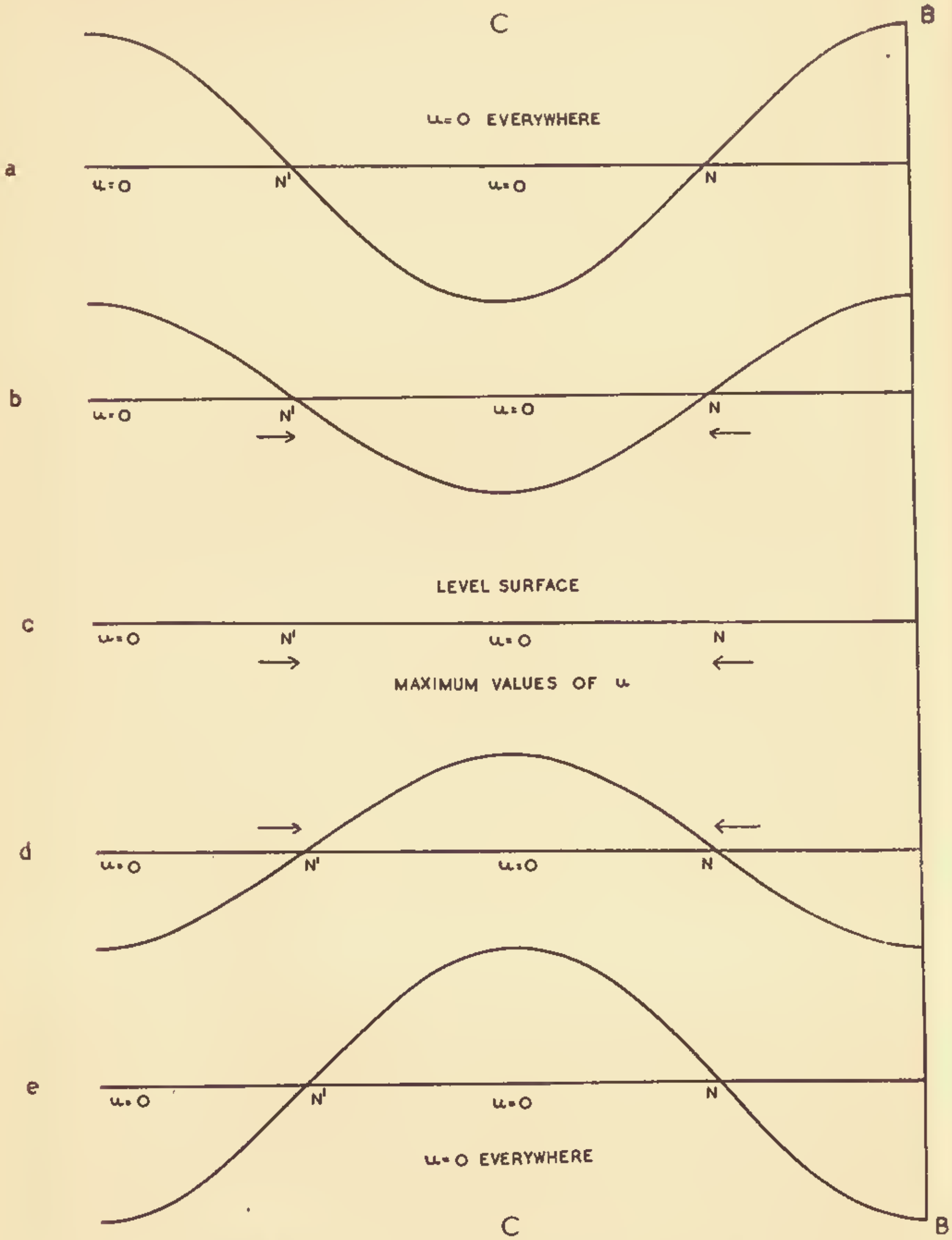

FIG. 18.3. Standing oscillations resulting from primary and reflected waves.

We are now in a position to draw the profiles of the resultant curves for the case of reflection at a barrier, by adding the elevations of the primary and reflected waves, and the results are shown in Fig. 18.3. We see that the water oscillates in such a manner that at certain places (N and N′) there is never any vertical movement at all, though the maximum horizontal velocities are encountered at such places. These points correspond to the nodal lines transverse to the channel. Further, the range of oscillation varies from zero at the nodes to the maximum values at the barrier and at those places whose distances from the barrier are multiples of half the wave-length. Also we have the noteworthy result that over large areas it is synchronously high or low water.

The relations between the elevation and the stream are clearly the same as in the case of the standing oscillations discussed earlier, in Art. 18.1, and in fact we have a standing oscillation with multiple nodes if the channel to the left is regarded as infinite.

18.4. Relations between standing oscillations and progressive waves

It was shown that the barrier at B in Fig. 18.2 was permissible because the resultant horizontal velocity was zero there. Similar reasons can be given to show that the horizontal velocity is always zero at places which are distant from B by multiples of half a wave-length. (These points are indicated in Fig. 18.3 by the statement $u = 0$.) We could therefore place barriers at any of these places, so that the two waves of Fig. 18.2 could be considered as moving backwards and forwards from one reflecting barrier to another. In this case we would get an enclosed sea corresponding to the basin discussed at the beginning of the chapter. We see now the reason why the length of the sea came out as equal to half the wave-length of a progressive wave travelling in a channel with the same depth as the enclosed sea.

In Art. 18.1, however, we only considered the simplest mode of motion, with only one nodal line, but the theory of reflected waves shows that we could have any number of nodal lines, and we conclude that for any given basin of length L, we must expect natural periods equal to

$$\frac{2L}{\sqrt{gh}} \div (1, 2, 3 \ldots)$$

but as a rule we only need to consider the first and second of these periods.

The application of these results to tidal movements will be discussed in Chapter XIX.

The relations between progressive waves, reflected waves, and standing oscillations are well brought out by a consideration of formulæ expressing harmonic motions. Let the time be denoted by t, and let the distance along a channel be denoted by x, taken positive in the direction from left to right. Then the expression

$$R \cos n(t - x/c) \quad . \quad . \quad . \quad . \quad . \quad (18.4a)$$

clearly represents an oscillation which has a period of $2\pi/n$ for any value of x, and high water takes place when

$$t = x/c \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.4b)$$

Hence the expression represents a progressive wave whose rate of propagation is c.

Similarly the expression

$$R \cos n(t + x/c) \quad . \quad . \quad . \quad . \quad . \quad (18.4c)$$

represents a progressive wave travelling in the negative direction. As written, both waves have the same amplitude R.

The velocities of the streams for the two waves are given by

$$U \cos n(t - x/c) \quad . \quad . \quad . \quad . \quad (18.4d)$$

$$\text{and} - U \cos n(t + x/c) \quad . \quad . \quad . \quad . \quad (18.4e)$$

where the velocities are measured in the direction of x increasing, so that the velocity of the wave travelling in the negative direction takes a negative sign. These formulæ express the ascertained fact that the velocity of the fluid in the direction of travel of each separate wave is a maximum at high water.

Now suppose that both waves exist together in the same channel. The elevation becomes

$$\mathrm{R}\cos n(t - x/c) + \mathrm{R}\cos n(t + x/c),$$

which is equal to

$$2\mathrm{R}\cos nt\cos x/c \quad . \quad . \quad . \quad . \quad . \quad (18.4f)$$

while the velocity similarly becomes

$$2\mathrm{U}\sin nt\sin x/c \quad . \quad . \quad . \quad . \quad . \quad (18.4g)$$

We see from the last expression that the velocity is zero when x/c is a multiple of π, so that the velocity is always zero at intervals of half a wave-length from any given node. The formulæ (18.4f) and (18.4g) apply equally to a channel closed at one end ($x = 0$) and to a channel closed at both ends ($x = 0$ being one end), provided that the length of the completely closed channel is a multiple of half a wave-length. The formulæ show that at high water ($\cos nt = 1$) or at low water ($\cos nt = -1$) the stream is everywhere zero, since $\sin nt = 0$.

18.5. Standing oscillations in shallow water

As in the case of the progressive wave, it is necessary to examine the effects of shallow water, when the amplitude of the oscillation is an appreciable fraction of the depth of the channel. It was shown in Art. 17.8 that a progressive wave is distorted in shape as the wave travels, owing to the rate of propagation being greater for the elevation than for the depression. It was pointed out that the depth h only appeared in the investigations when it was associated with the elevation y, in the form $(h + y)$, for the reason that the flux across a sectional plane is proportional to the area of the plane and this area is proportional to $(h + y)$. It was also pointed out that a harmonic variation of the elevation would tend to be associated with a harmonic variation of the flux, and as the flux is proportional to the product of the velocity u and the sectional area $b(h + y)$ then the stream could not be also expressed by a simple harmonic. Then it was remarked that any complexity in the stream would inevitably lead to a similar complexity in the elevation owing to the inter-relations between the elevation and the stream.

Similar considerations can be applied to standing oscillations. For any given state of tide the harmonic law of variation would give an equal and opposite value for the elevation half a period later. The rate of stream would also be reversed in sign and would retain the same magnitude if the channel were very deep, but in shallow water the velocity would be greater for negative elevations of tide than they would be for equal positive elevations. Therefore, even if the elevation were to follow a simple cosine law, the streams could not do so, and similarly, even if the streams could be expressed by a simple harmonic term, the same could not be done for the elevation. Therefore, as there is an interaction between elevation and stream, it follows that neither can be expressed by a simple harmonic term, except when the water is very deep. In shallow water there must be distortion, so that, for example, as was shown in Chapter VIII, a semidiurnal tide will be accompanied by quarter-diurnal tides and higher species of shallow-water tides.

CHAPTER XIX

TIDES AND TIDAL STREAMS IN GULFS, SEAS, ESTUARIES, AND OFFSHORE

19.1. Tides as forced motions

In the theory of progressive waves and standing oscillations it was supposed that the water had been disturbed and that we were only concerned with its subsequent movements, on the assumption that the original cause of disturbance had ceased to operate. It was shown that the vibratory motion involved an interchange of energy from the potential to the kinetic form and *vice versa*, and that when energy has been communicated to the system the vibrations continue indefinitely in the absence of friction. In whatever way the water has been initially disturbed it will thereafter move according to the conditions imposed by the dimensions and shape of the basin or channel.

Such motions, which continue to take place after the initial disturbance has ceased to operate, are called *free* motions.

The tides, though, are caused by forces of astronomical origin which are continually acting, and therefore the tidal motions experienced on the earth are *forced* motions. We are not yet in a position to discuss the problem of the general response of the oceans to the tide-generating forces, but we can take the oscillations in an ocean as the immediate source of tidal oscillations in gulfs and estuaries. A comparison of the areas of the indentations of the coast of an ocean with the area of the ocean itself shows that the tides in gulfs must be regarded as off-shoots of the large-scale oscillations in the oceans, and they are said to be *maintained* by the tides in the oceans.

The relations between the free and forced oscillations are of very great interest and importance, and the analogy of a pendulum will illustrate one of the points of interest. Suppose that a pendulum is swinging freely in a period which is determined by the length of the pendulum, its *natural* period. The arc through which it swings is dependent upon the initial supply of energy, whether communicated in the potential form by elevating the bob of the pendulum, or in the kinetic form by an impulse to the bob. Suppose that an impulse is given at the top of the swing; then additional kinetic energy is imparted to the pendulum so that when all the energy is transformed to the potential form by the pendulum rising to the top of its swing again it will be higher than it was on previous swings. If another impulse is then given, a further supply of energy is imparted and the arc of swing will increase. A succession of such small impulses will ultimately increase the arc beyond the limits of true oscillation and a state of *resonance* is said to occur. Similarly, in connection with tides, if the dimensions of a gulf or sea are such that the period of a free oscillation is equal to that which is maintained in the ocean, there is a liability to resonance.

If, however, the impulses are not given in the proper period, that of the free movement of the pendulum, they may increase or decrease the arc of swing temporarily, but there will not be any cumulative effect.

19.2. Tides in a gulf

The relations between the tides in a gulf and those in the ocean may be derived from the theory of reflected waves given in Art. 18.3. The standing oscillations for a channel of uniform sections are illustrated in Fig. 18.3, and it was supposed that waves coming along the channel were reflected at the barrier at B. But the theory of Art. 18.2 showed that what is necessary to the continuance of the motion in the channel between any section and the barrier is that the elevation and the stream at

that section shall have certain relations. If then these relations are maintained, the motions will continue in the same way as though in actuality systems of waves existed to an infinite distance. The motions between any sectional plane and the closed end of the channel are indifferent to the circumstances outside this area, so long as the proper supplies of energy, kinetic and potential, are forthcoming.

In order to elucidate the relationship between the tides in a gulf and the parent tides in the ocean, we shall first consider a simple case in which there are no tidal streams at any time across the entrance to the gulf. The possibility of such conditions follows from a reconsideration of the theory of reflected waves as given in Art. 18.3. From Fig. 18.3 we see that at the point C (in the channel which is closed at the end B) there is always zero velocity of stream. The length of the portion of

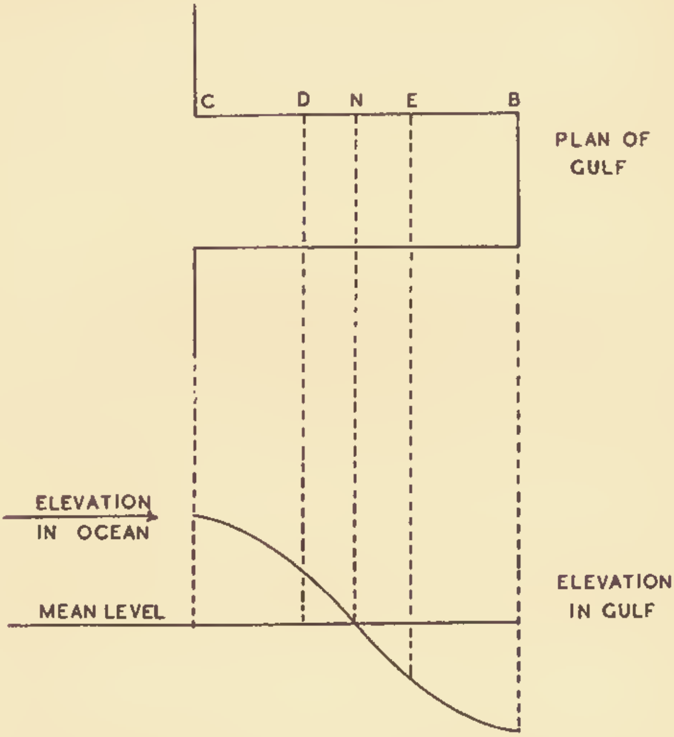

FIG. 19.1. Tides in gulf with no tidal streams at entrance.

the channel from C to B is equal to half the wave-length of a progressive wave travelling in a channel of the same uniform depth as that of the channel CB. Let part of Fig. 18.3e be redrawn as in Fig. 19.1 for the portion CB. We know that an oscillation can take place in CB such that there is a rise and fall of the surface at C without there being any horizontal velocity of water. If, therefore, this elevation changes in synchronism with that of the ocean outside the gulf then the two motions, in the gulf and in the ocean, take place without any interference between one another. The gulf is receiving no kinetic energy from the ocean and it has maintained for it the proper condition of elevation at the entrance. Hence the tide in the gulf is such that at the entrance it is the same as in the ocean just outside the gulf, and the tide inside the gulf is then given by the theory of Art. 18.3.

We see, therefore, that the shape of the profile of the elevation of tide in the gulf is dependent only on the physical characteristics of the gulf and not upon anything outside it. Thus if the channel has a rectangular shape and a constant depth (h) there would be a nodal line half-way along the gulf, exactly as in Fig. 18.3, and the relation between the period of the oscillation and the length of the gulf would be the

same as is given in (18.2i), so that the length of the gulf (L) for this case must be equal to

$$\tfrac{1}{2}\,T\sqrt{gh} \qquad . \quad . \quad . \quad . \quad . \quad (19.2a)$$

Since the period T is fixed by that of the tidal oscillation in the ocean this expression gives the length of the gulf for the case illustrated.

Now suppose that the gulf is shorter than in the case just considered, and that the entrance is at D in Fig. 19.1. The gulf will require a supply of energy at D from the ocean in order to maintain the tides in the gulf, and there will now be tidal streams across the entrance at D. The elevation at the entrance must be the same as in the ocean just outside. We see at once from Fig. 19.1 that the range of tide will now be

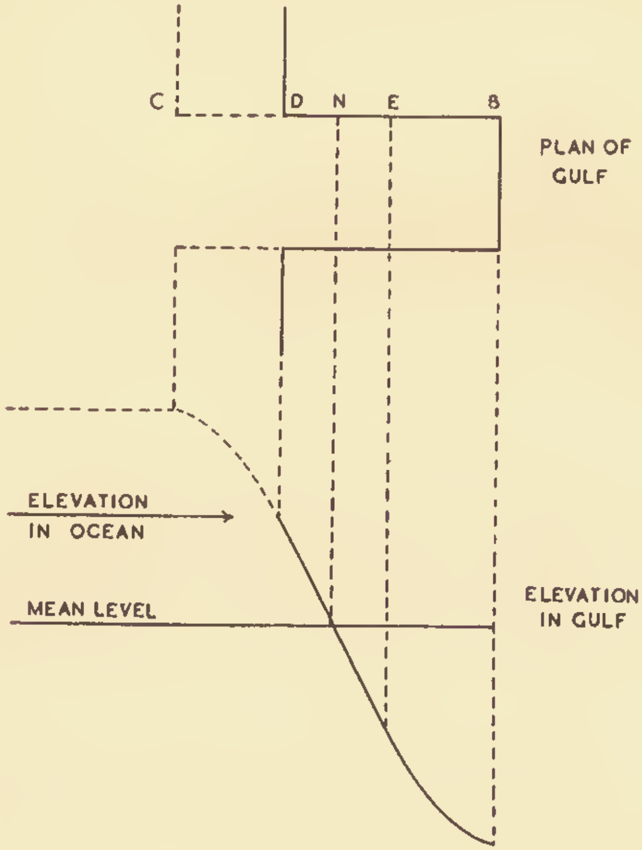

FIG. 19.2. Tides in gulfs which are longer than a certain critical length.

greater at the barrier B than at the entrance D if we have the oceanic tide the same as before. Hence we have to enlarge the vertical scale as in Fig. 19.2. The dotted lines in this figure show the connection with Fig. 19.1.

The important result has been obtained that if the gulf is somewhat shorter than half the length of a progressive wave in a channel of the same depth, the range of tide at the head of the gulf is greater than at the entrance to the gulf, and it is low water at the head of the gulf when it is high water at the entrance.

The gulf will affect the tides in the ocean to a certain extent owing to the tidal streams set up across the entrance to the gulf, but the effects will be small if the gulf is small compared with the ocean, and they will be practically localised near the entrance to the gulf. There will be certain adjustments of elevation and stream until the conditions imposed by the gulf are quite satisfied.

We see, therefore, that as the gulf diminishes in length from CB to NB there will be an increase in the ratio of tidal range at the head of the gulf to the tidal range at the entrance, and it is clear that when the critical length NB is approached a state of resonance will take place. According to this simple theory the tides would then become infinitely large, but other considerations would then have to be taken into account, such as the effect of the gulf on the oceanic tides, the effects of friction, and so forth.

The critical lengths of a gulf, for various depths, are given in the following table, for tides of periods of 12 or 24 hours.

TABLE 19.1

Critical Length of Gulf

Depth in Fathoms	Critical Length in Nautical Miles. Period = 12 Hours.	Period = 24 Hours.
50	175	349
100	247	494
200	349	698
500	552	1104
1000	780	1561
2000	1104	2207

This table has been derived from the formula

$$L = \tfrac{1}{4} T\sqrt{gh} \qquad \qquad (19.2b)$$

since the length is only one-half of the length of the gulf given by (19.2a). Alternatively, it may be readily derived by multiplying by three and six, respectively, the values of $\sqrt{gh}$ given in Table 17.1.

The table given above is of great importance, for it is evident that the depths and lengths are of the order experienced in nature. The applications of the results of this article, however, will be made in later chapters.

When the length of a gulf is less than the critical length there is no nodal line separating regions of low water from regions of high water, so that it is high water in the gulf when it is high water in the ocean at the entrance. This can be verified from Fig. 19.3 in which the profile at high water in the gulf EB is drawn for that gulf and indicated by dotted lines for the gulf CB. If this is compared with Fig. 19.1 it will be clear that we have to draw the profile to correspond with the elevation at the entrance in each case. We see that the tide at the head of the gulf EB has a greater range than at the entrance.

To sum up, we have shown that :—

(*a*) the tides in a gulf are standing oscillations ;

(*b*) if the gulf is shorter than a certain critical length the tide throughout the gulf is at high water at the same moment as the oceanic tide, and the range of tide increases from the entrance of the gulf to the head of the gulf ;

(*c*) if the gulf is longer than the critical length the tide at the entrance is the same as the oceanic tide, and the range decreases through zero at a nodal line and thereafter increases until the head of the gulf is reached, where the tidal range is greater than at the entrance, and it is low water at the head when it is high water at the entrance to the gulf.

Very little needs to be said about the streams in a gulf, because it is evident that the streams are related to the elevations in exactly the same manner as in the general case of standing oscillations. The stream has its maximum rate at half tide when the surface of the water is flat throughout the gulf, and it flows towards the place where the tide is about to rise.

19.3. Forced oscillations in an enclosed sea or ocean

The maintenance of large oscillations in a gulf, as discussed in the previous article, is really dependent upon the equality of the periods of the oceanic tides and

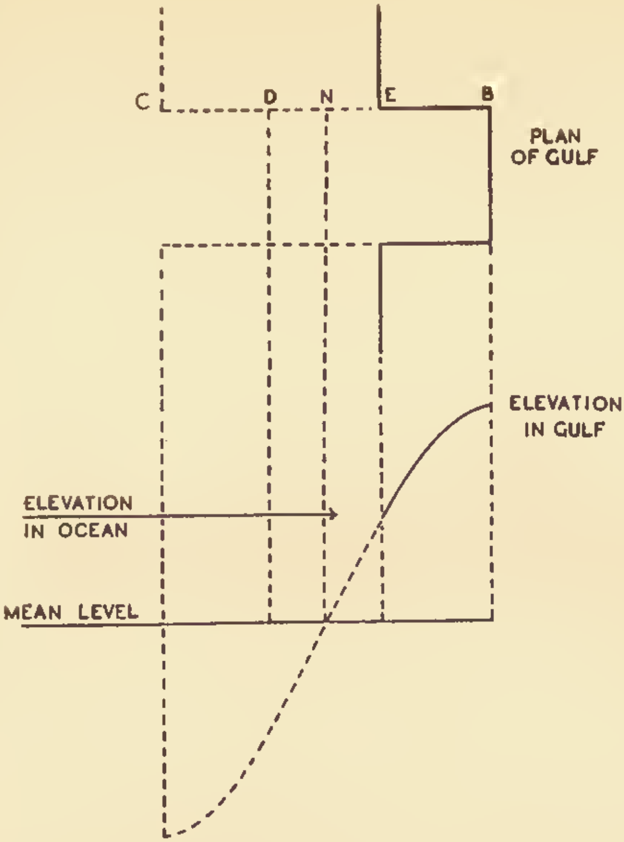

FIG. 19.3. Tides in gulfs which are shorter than a certain critical length.

the free oscillations proper to the gulf. We have supposed that the tides in a gulf are only influenced by the oceanic tides and the direct effects of the tide-generating forces were ignored. Before we can consider these we must consider an enclosed sea, in which the maintenance of the forced tidal oscillations must be due to the direct effects of the tide-generating forces. If these forces cause impulses recurring in phase with the natural free oscillations proper to the sea, then they will maintain a forced oscillation of very large amplitude. For a rectangular sea of length L and uniform depth h, we showed in Art. 18.2 that the period (T) of the free oscillation is given by formula (18.2i) as :—

$$T = \frac{2L}{\sqrt{gh}}$$

TABLE 19.2

Critical Lengths of Enclosed Sea or Ocean

Depth in Fathoms.	Critical Length in Nautical Miles. Period = 12 Hours.	Period = 24 Hours.
50	349	698
100	494	987
200	698	1396
500	1104	2207
1000	1561	3122
2000	2207	4415

It will be noticed that these values are exactly twice those given in Table 19.1 for the critical length of a gulf.

It may be asked, How is it that the critical lengths of a gulf and enclosed sea are not equal, since standing oscillations occur in both? The answer is that we have really been dealing with two different types of problems. In the case of the gulf we assumed that all the tidal action was due to the ocean outside, and that none was generated inside the gulf, whereas in the case of the sea all the tidal action was generated within the sea by the action of the astronomical forces. There would be a tendency to resonance within a gulf of the same dimensions as the sea discussed above, due to the action of the tide-generating forces within the gulf, but some allowance would need to be made for the effect of the open end, and generally speaking the gulfs specified in Table 19.1 are more likely to occur on the earth than gulfs of the order of dimensions given in Table 19.2.

19.4. Tides in converging estuaries

Hitherto it has been supposed for the sake of simplicity that the tidal movements have been taking place in channels of uniform width and depth, but as it is more general for the width and depth to decrease up an estuary it is desirable to consider how the elevation and stream are affected by the changes in the cross-sectional area. In general terms it is clear that if the area diminishes the streams must increase in order to convey the same amount of volume and energy as before. This necessarily involves an increase in the elevation also, but a precise relation between the changes in area and the changes in the elevation and stream can only be obtained by mathematical methods. It is possible, however, to consider the case of a progressive wave, moving in a channel whose cross-sectional area slowly changes, in a very elementary manner.

Suppose, as usual, that the breadth of a channel is given by b, the depth by h, and that these are not constant, though we shall continue to suppose that the walls of the channel are vertical. Let the wave-length be denoted by l, and consider a portion of fluid between two cross-sectional planes separated by a distance l/m, where m is an integer, so that there are m such portions per wave-length. We shall suppose that the two sections we are interested in will be located at some definite part of the wave.

In Art. 17.6 we obtained expressions for the kinetic and potential energies of a wave and it is easy to deduce that the potential energy of the fluid located between the two sectional planes, if the fluid has density ρ, is given by

$$\tfrac{1}{2}\rho g b y^2 \frac{l}{m} \qquad \qquad (19.4a)$$

where y is the mean value of the elevation between the two planes, which are supposed to be very near together.

Similarly the kinetic energy of the water between the two planes is given by

$$\tfrac{1}{2}\rho h b u^2 \frac{l}{m} \qquad \qquad (19.4b)$$

where u is the mean value of the rate of stream across the two planes.

It was shown in Arts. 17.4 and 17.7 that

$$hu = cy \qquad \qquad c^2 = gh$$

and on substituting from these equations in (19.4b) we find that the kinetic energy localised between the two planes is equal to the potential energy, a result which was proved earlier for the whole wave. Hence we get the total energy of this element of volume is equal to

$$\rho g b y^2 \frac{l}{m} \qquad \qquad (19.4c)$$

or to

$$\rho h b u^2 \frac{l}{m} \qquad \qquad (19.4d)$$

Now the energy of the wave per wave-length must remain unaltered, and if the changes in the channel are very gradual it follows that the energy of any definite fraction of the volume of the wave at any definite phase of it must also remain unaltered. Also the period (T) of the wave must remain the same, and since the wave-length l is equal to cT, where $c = \sqrt{gh}$, it is clear that from (19.4c) we have

$$gby^2\sqrt{gh} = \text{a constant}$$

whence we get

$$hb^2y^4 = \text{a constant} \quad . \quad . \quad . \quad . \quad (19.4e)$$

and in a similar fashion, from (19.4d)

$$h^3b^2u^4 = \text{a constant} \quad . \quad . \quad . \quad . \quad (19.4f)$$

As illustrations of the use of these formulæ, suppose that at places P_1 and P_2 the values of the elevation, breadth and mean depth of water are denoted by y_1, b_1, h_1 and y_2, b_2, h_2 respectively. Suppose also that at P_2 the mean depth is only one-quarter of that at P_1, and that the breadth is only one-half that at P_1, so that

$$h_1 = 4h_2, \quad b_1 = 2b_2$$

and therefore, from the relation

$$h_2{b^2}_2{y^4}_2 = h_1{b^2}_1{y^4}_1$$

we get

$${y^4}_2 = 16{y^4}_1$$

or

$$y_2 = 2y_1$$

Similarly we get

$${h^3}_2{b^2}_2{u^4}_2 = {h^3}_1{b^2}_1{u^4}_1$$

and

$$u_2 = 4u_1$$

In general terms it is clear from the formulæ that the elevation is more susceptible to changes in the width of the channel than to changes in the depth, whereas the stream is more susceptible to changes in depth than to changes in the width.

What is true of a progressive wave is true also of a reflected wave, and therefore a standing oscillation changes in elevation and in the rate of stream according to the above laws.

The theory given above is only accurate if the changes in sectional area are small within a wave-length. In actuality this condition may not be fulfilled, in which case there will be a continuous reflection of waves at all points of the bed and sides of the estuary, instead of only at the barrier at the head of the estuary. But the general conclusions obtained above are of value in indicating the character of the exaggeration of oceanic tides which takes place in shallow seas and in estuaries. It will be noticed that we have two distinct causes of such exaggeration,

(1) by reason of the tendency to resonance
and (2) by reason of the changes in the area of the cross-section of the channel.

It will be apparent that the latter is not dependent upon the period of oscillation, so that diurnal and semidiurnal oscillations will be equally magnified, whereas in the former case the period of oscillation must be related to the free periods of the estuary for exaggeration to take place, and it is unlikely that both species of tides will be affected to the same degree.

CHAPTER XX

FORCES DUE TO THE EARTH'S ROTATION (OR GYRATION)

20.1. Centrifugal forces

In the three preceding chapters we have been concerned with the motions of water as affected by the physical characteristics of the basin in which the motions take place.

It may be remembered that we found it convenient to consider a progressive wave by using the artifice of moving with the wave, and so to consider a state of relative motion, but as all motions are relative motions because they take place on a moving earth, we are called upon to examine the effects of the earth's rotation. We shall show that forces are brought into existence by the rotation of the earth which cause rotatory changes in tides and tidal streams.

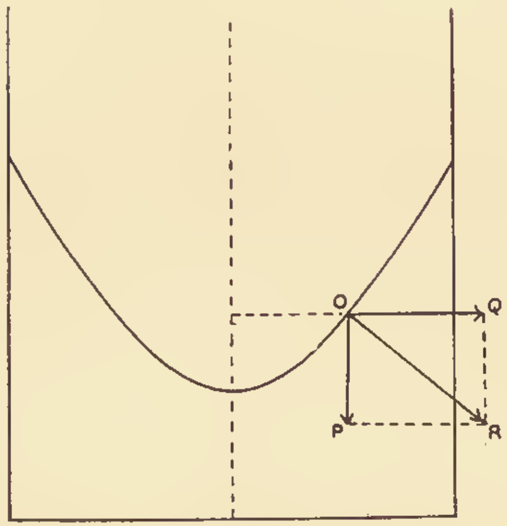

FIG. 20.1. Surface of equilibrium of a revolving fluid.

To avoid confusion, we shall only use the word *rotation* in future in connection with the tides and tidal streams, and the word *gyration* will be used for the rotation of the earth on its axis, leaving the word *revolution* for the general cases of rotating bodies.

It is well known that a body in motion round a circle tends to be deflected from the curve unless there are counteracting forces or physical constraints, and a stone in a sling gives an obvious example of forces associated with the motion of revolution, but as we are more interested in the effects upon fluids we shall consider in detail the case of a fluid in a revolving vessel.

It is a matter of common knowledge that if a quantity of water is contained in a revolving vessel, then the surface of the water becomes concave upwards, being depressed in the middle and elevated in an increasing degree towards the edge. Since the effect is to throw water away from the centre it is evident that there is brought into existence by the act of revolution a force outwards from the axis of revolution. It is this force, which is called *centrifugal force*, which is operative in building up the surface gradient, and this operation continues until equilibrium is

reached, when every particle of fluid is revolving at a steady rate. Under these circumstances the surface of the fluid is at right angles to the resultant force, which is composed of the gravitational and centrifugal forces. Fig. 20.1 illustrates the combination of the forces which are respectively represented in direction and magnitude by the lines OP and OQ, the resultant being represented by OR. The curved line, which is the representation of the fluid surface intersected by a plane through the axis of revolution, is at right angles to the resultant force.

It will be shown mathematically in Art. 20.3 that the centrifugal force under steady conditions of revolution varies as the distance from the axis of revolution, so that in the steady state each particle is revolving round the axis at a velocity appropriate to its distance from the axis. We note that though any particle may appear to be stationary relative to its surroundings and to the containing vessel (we shall refer to these as the apparent conditions) yet in actuality it has a definite velocity in space. Now suppose that the velocity in space is increased, say by an impulse, so that the particle of water, whicb has been stationary relative to the revolving vessel, is given apparent velocity in the direction of revolution. The conditions of equilibrium are now disturbed and the particle moves outwards until it finds a position of apparent rest appropriate to its real space velocity. If, however, the particle is given an impulse in the opposite direction to that of revolution, then the centrifugal force due to its real velocity is decreased and it moves inward until it finds a position of equilibrium. It will be noted that the work done on the particle in giving it enhanced velocity in effect lifts it against gravity in the case considered.

The principal point revealed by the above discussion is that a particle disturbed from a position of equilibrium under conditions of revolution experiences forces upon it which direct it to the right or left of its path according to whether the apparent motion is in the direction of revolution or is opposed to it.

When we consider the centrifugal forces in water on the earth's surface, we see that we can apply similar methods of investigation. The water will tend to set itself everywhere so that its surface is at right angles to the resultant of the gravitational and centrifugal forces, but in tidal theory we are not very much concerned with the actual surface under steady conditions of gyration. We are, however, greatly concerned with the effects of disturbing the conditions of equilibrium by imparting apparent velocity to a particle, and we proceed to discuss the forces set up.

20.2. Gyroscopic forces

We shall consider the problem of disturbances from the steady conditions existing under gravitational and centrifugal forces by two methods, the first for disturbances which give apparent velocity in an eastward or westward direction, and the second in which the apparent velocity is in a northward or southward direction.

In the first case, it is evident that as the velocity in the direction of rotation is increased, then the centrifugal force on the particle is greater than that due to its distance from the axis, and it will therefore tend to move still further away from the axis of gyration ; that is, it is directed towards the equator. If, however, the apparent velocity imparted is to the west, then its real velocity is diminished and the centrifugal force upon it is less than that due to its position, so it must move, if possible, to a smaller distance from the axis of gyration ; that is, it must move towards the pole. In either case, therefore, it moves to the right of its apparent velocity on the earth's surface in the northern hemisphere, but in the southern hemisphere a particle with motion in longitude will be deflected to the left of its motion.

It is evident that while the centrifugal forces we have considered are in the plane of the circle of latitude we are really concerned with the tractive forces along the earth's surface, so that the difference between the centrifugal force due to the particle's position and that due to its real velocity has to be resolved along the earth's surface. This component of the differential centrifugal forces is called the *gyroscopic force* on the particle. At the equator, the centrifugal forces are at right angles to the earth's surface, and there is no component along the earth's surface so that at

the equator the gyroscopic forces are zero, corresponding with the change in deflection of motion in the two hemispheres north and south of the equator.

We have so far only dealt with the gyroscopic forces due to motion in longitude; we shall now consider the second case, that of motions in latitude. It is well known that a wind blowing north is deflected to the right in the northern hemisphere, and in fact this is the explanation of the directions of the trade winds. Similarly a particle of water will be deflected to the right in the northern hemisphere and to the left in the southern hemisphere. The reason for this is that, in the northern hemisphere, a particle in one latitude has a greater velocity in the direction of gyration than one in a more northerly latitude, and that if it is disturbed from its equilibrium position and carried north, then it will retain an eastward component of velocity (in the absence of friction) which is greater than the velocity appropriate to the position further north. In other words, it is carried to the east by the excess of velocity in an east direction. If, however, it is given a motion to the south, then its east component of velocity will be smaller than that proper to the new latitudes and it lags behind, and so appears to be carried to the west. In both cases it is therefore diverted to the right of its motion in the northern hemisphere, but in the southern hemisphere, the diversion is to the left of its motion.

It is shown in Art. 20.4 that the force which causes a particle with a movement in latitude to be deflected to the right of its motion in the northern hemisphere is exactly the same in magnitude as the force which affects a movement in longitude with equal velocity. Therefore, for movement in any direction, what is true of each component is true of the resultant velocity, and the gyration of the earth causes *gyroscopic forces* which are directed to the right of the motion in the northern hemisphere, and to the left of the motion in the southern hemisphere. These forces only arise where there is a velocity relative to the earth's surface (or to the bed of the ocean) and they are proportional in magnitude to the velocity of the particle. It is shown in Art. 20.4, though it is a necessary consequence of the change in direction at the equator, that the gyroscopic forces increase with the latitude north and south of the equator, being, in fact, proportional to the sine of the latitude.

The mathematical expression for the gyroscopic force, as obtained in Arts. 20.3 and 20.4 is

$$2mV\omega \sin l \qquad \text{(20.2a)}$$

where m is the mass of the particle in motion
V is its velocity relative to the earth
ω is the angular velocity of the earth
l is the north latitude of the place where the motion occurs.

The general conclusions obtained above are obviously in conformity with the indications of the formula.

*20.3. Mathematical theory of centrifugal forces

Before considering the mathematical details of the theory of centrifugal and gyroscopic forces, we shall refer to the theory of motion under a force. A force causes a change in velocity and so it is measured by the acceleration it produces. It is shown in Art. 17.9 that a particle under gravity acquires a velocity gt in time t, where g is the acceleration or "the coefficient of gravitational force," and that the distance travelled in time t is equal to $\frac{1}{2}gt^2$. Now what is true of gravitational force is true of any similar force, and if f is the acceleration due to a force F then the distance travelled in time t under its influence is $\frac{1}{2}ft^2$. We can thus define the force F by the acceleration f, and refer to the latter as "the coefficient of the force F."

In order to compute the centrifugal force, we shall suppose that a particle is moving in a circular path of radius r with uniform velocity v. Let a small portion of the path (Fig. 20.2) be denoted by AB, and let BD be a perpendicular upon AD, the tangent at A. Let O be the centre of the circle and let the angle AOB be denoted by θ. Let f be the coefficient of the centrifugal force, and let t be the time taken to travel from A to B. Now the particle would have been found at D instead of B if

* See par. 1, page vii.

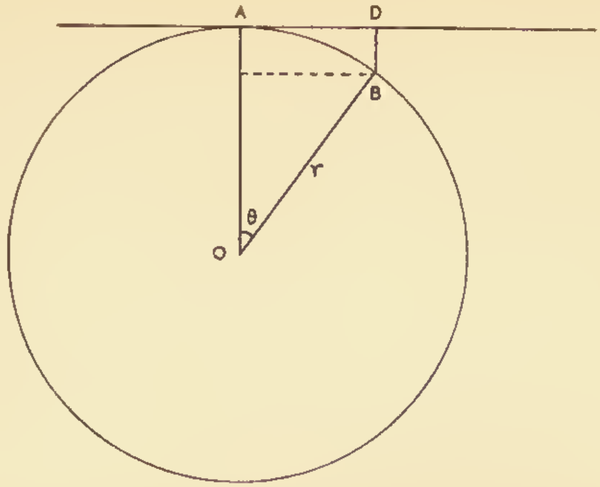

FIG. 20.2. Calculation of centrifugal forces.

it had been released from constraint at A and therefore BD is the distance through which the particle would have moved under the centrifugal force, so that

$$\mathrm{DB} = \tfrac{1}{2} ft^2 \quad \text{. . . . . . . (20.3a)}$$

We also have

$$\mathrm{AD} = vt \quad \text{. . . . . . . (20.3b)}$$

and if θ is very small, we have approximately

$$\mathrm{AD} = r\theta \quad \text{. . . . . . . (20.3c)}$$

$$\mathrm{DB} = r - r\cos\theta = \tfrac{1}{2} r\theta^2 \quad \text{. . . . (20.3d)}$$

whence we readily reduce

$$\mathrm{DB} = \tfrac{1}{2}\,\frac{(\mathrm{AD})^2}{r} = \tfrac{1}{2}\,\frac{v^2t^2}{r} \quad \text{. . . . . (20.3e)}$$

and, therefore, from (20.3a)

$$f = \frac{v^2}{r} \quad \text{. . . . . . . (20.3f)}$$

The centrifugal force on a particle of mass m is thus equal to

$$\frac{mv^2}{r} \quad \text{. . . . . . (20.3g)}$$

If ω is the angular velocity, then

$$v = r\omega \quad \text{. . . . . . (20.3 h)}$$

and the centrifugal force may be written in the alternative form

$$mr\omega^2 \quad \text{. . . . . . (20.3i)}$$

***20.4. Mathematical theory of gyroscopic forces**

We shall consider the problem of disturbances from steady conditions, existing under gravitational and centrifugal forces, by two methods, the first for disturbances in longitude and the second for disturbances in latitude, both in the northern hemisphere, unless otherwise stated.

Suppose that a particle of mass m is moving under steady conditions of gyration in latitude l, so that it is describing a circle of radius r. Then its velocity is $r\omega$ and the centrifugal force upon it is $mr\omega^2$, as shown in the preceding article. To an observer moving with the earth, the particle is apparently at rest. Let now a velocity V in

* See par. 1, page vii.

the direction of rotation be imparted to it, so that its real velocity in space is $r\omega + V$, which is equal to

$$r\left(\omega + \frac{V}{r}\right) \qquad . \quad . \quad . \quad . \quad . \quad (20.4a)$$

so that the angular velocity has been changed from ω to $\left(\omega + \frac{V}{r}\right)$. Therefore the apparent centrifugal force due to its real motion is changed to

$$mr\left(\omega + \frac{V}{r}\right)^2 \qquad . \quad . \quad . \quad . \quad . \quad (20.4b)$$

which is equal to

$$mr\omega^2 + 2m\omega V + \frac{mV^2}{r} \qquad . \quad . \quad . \quad . \quad . \quad (20.4c)$$

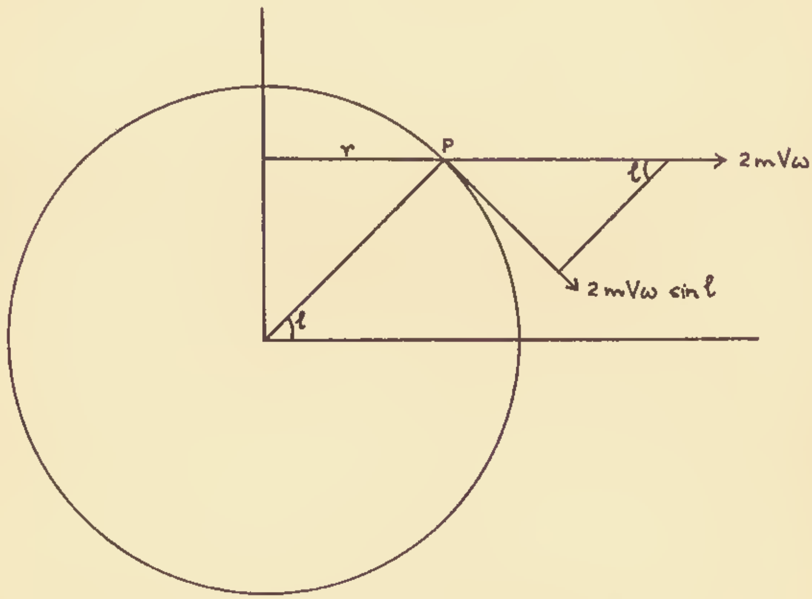

FIG. 20.3. Calculation of gyroscopic forces.

The third part of this expression is negligibly small because of the divisor r, and the first part is the normal centrifugal force. The middle term is the force brought into existence by the movement of the particle over the earth's surface. We are not interested, of course, in the component of this force at right angles to the earth's surface because its only effect is to alter the apparent force of gravity, but the horizontal component of the force along the earth's surface (Fig. 20.3) will move the particle towards the equator (*i.e.*, to the right of its motion). The magnitude of this tractive force, which is called gyroscopic force, is

$$2mV\omega \sin l$$

so that the coefficient of gyroscopic force is

$$f = 2V\omega \sin l \qquad . \quad . \quad . \quad . \quad . \quad (20.4d)$$

We shall now consider the problem of the gyroscopic forces on a particle moving along a line of longitude. Firstly, we shall remark that if the radius of the earth is a, and if its angular rate of gyration is ω, then a particle in latitude l has a velocity given by $\omega a \cos l$. This velocity decreases with the latitude, and therefore if a particle

moving steadily at this rate is given an impulse northward, it will find itself travelling to the east at a quicker rate than particles moving steadily with the earth. Hence it is deflected to the east, that is, to the right of its motion relative to the earth.

Let the initial latitude be $(l - \theta)$ and let it be in latitude $(l + \theta)$ after a time t. Then it has moved eastwards relatively to points on latitude $(l + \theta)$ which were initially in the same longitude as itself, by the amount

$$\omega ta \cos (l - \theta) - \omega ta \cos (l + \theta) \quad . \quad . \quad . \quad (20.4e)$$

which is equal to

$$2\omega ta \sin l \sin \theta$$

and if θ is small, this can be written as

$$2\omega ta\theta \sin l \quad . \quad . \quad . \quad . \quad . \quad . \quad (20.4f)$$

In the same interval of time t, the particle has travelled north with a velocity V, and clearly it has moved through a distance

$$Vt = 2a\theta \quad . \quad . \quad . \quad . \quad . \quad (20.4g)$$

Hence the deflection eastwards given by (20.4f) can be written as

$$V\omega t^2 \sin l \quad . \quad . \quad . \quad . \quad . \quad (20.4h)$$

Now it was stated in Article 20.3 that a particle moving under the influence of an acceleration f, will in time travel a distance equal to $\frac{1}{2} ft^2$, and we may write

$$V\omega t^2 \sin l = \tfrac{1}{2} ft^2$$

Therefore the coefficient of the gyroscopic force which may be regarded as causing the deflection is

$$f = 2V\omega \sin l \quad . \quad . \quad . \quad . \quad . \quad (20.4i)$$

exactly as in the case of east and west motion.

Since any velocity can be resolved into components to the east and to the north, it follows that in general the gyroscopic force acting on a particle of mass m moving with velocity V relatively to the earth's surface is

$$2mV\omega \sin l$$

and that this force is directed to the right of the motion in the northern hemisphere, to the left in the southern hemisphere, is zero on the equator and a maximum at the poles.

(It will be noted that the gyroscopic forces have been discussed by two different methods without any obvious logical connection. The method of proof by mathematicians is more general and logically consistent but it is rather too abstruse for this Manual.)

CHAPTER XXI

ROTATORY TIDAL STREAMS

21.1. The phenomenon of rotatory streams

THE preceding chapter shows that gyroscopic forces will cause deflections of particles in motion, and we shall now consider the cumulative effects of these and other forces on periodic motions. We know from observations that tidal streams in most cases do not simply flow to and fro in one direction, but they change direction and " rotate " in the tidal period. Generally speaking, the velocity of the stream changes as well as the direction, though it is not unknown for the velocity to remain practically unchanged as the direction changes. In the general case, if the velocity and direction of the stream are exhibited graphically hour by hour then a curve joining the ends of the " vectors," which denote the direction and velocity of the stream, will take an elliptical form, and in the special case where the velocity is constant for all directions of stream the diagram will take the shape of a circle.

Several causes can be given for the phenomenon of rotatory streams, and we shall deal with these in the following order :—

(*a*) rotation directly due to the tide-generating forces ;
(*b*) rotation due to the earth's gyration ;
(*c*) rotation due to the combination of two standing oscillations ;
(*d*) rotation due to the combination of a progressive wave with a standing oscillation ;
(*e*) rotation due to shelving coasts.

Strictly speaking, all these can be considered as coming under the general theory of (*c*), but for convenience we shall classify the causes of the phenomenon in the above way. It is necessary to point out that many of these causes may be acting at the same time ; thus in an ocean or very large sea the tide-generating forces will have direct effects and so will the gyroscopic forces. In smaller seas and broad gulfs the direct effects of the tide-generating forces will be small, the tidal streams will be maintained from the general oceanic tides and the tides experienced may be represented by combinations of standing oscillations, but the gyroscopic forces will also be effective, and so on.

As it is impossible to consider in a simple way all these causes when acting together it is necessary to consider them separately. It will be found that the results vary, and that sometimes the rotation will be in the clockwise direction and sometimes in the anti-clockwise direction. Taking the causes as acting together the separate effects may thus reinforce one another or annul one another, and this must be borne in mind.

21.2. Rotatory streams due to the tide-generating forces

It was shown in Chapter III that the instantaneous values of the tide-generating forces had easterly and northerly components, and it was deduced that if a particle responded to these forces it would show evidence of forces which varied in the tidal period through the points of the compass. The formulæ of Art. 3.4 show that the variations in the components of the lunar tractive forces at a place X are proportional to the following expressions, in which l is the north latitude of the place, d the north declination, Z the longitude of the place east of the meridian of lower transit of the moon.

	North Component.	East Component.
Diurnal tractive force . .	$-\cos 2l \sin 2d \cos Z$	$\sin l \sin 2d \sin Z$
Semidiurnal tractive force .	$-\frac{1}{2} \sin 2l \cos^2 d \cos 2Z$	$-\cos l \cos^2 d \sin 2Z$

Consider now the semidiurnal forces. The values of sin $2l$ and cos l are positive in the northern hemisphere for all values of the latitude l. When the hour is zero ($Z = 0$) the north component of force is negative, and the east component is zero. Hence the force is directed to the south. At hour 1, with $Z = 30°$, the east component is negative and there is also a south component, so that the direction of the resultant force is somewhat to the west of south. If we proceed for other hours we see that the direction of the semidiurnal force rotates in a clockwise manner throughout the northern hemisphere. In the southern hemisphere (in which l is negative, sin $2l$ is negative, and cos l is positive), the direction of rotation of the forces is anti-clockwise.

The diurnal forces vary in a more complex manner, since sin $2d$ may be positive or negative and cos $2l$ changes sign when $l = 45°$, but when d is positive and l is less than 45° then with $Z = 0$ the direction of force is to the south; hence we see that the diurnal forces in this zone rotate in an anti-clockwise manner. There are four zones, each covering 45° of latitude, and at the latitudes 45° N., 0°, 45° S., the rotation changes from an anti-clockwise direction to a clockwise direction, or *vice versa*.

We expect, somewhere or other, similar variations in the velocity of the water, but we cannot deal with the whole problem because we have not considered the hydraulic forces brought into operation as the tidal flow builds up elevations which set up counter-acting forces and velocities.

The above investigation deals separately with the diurnal and semidiurnal forces, but it was shown in Art. 3.3 that the necessity for the resolution of the tractive forces into components arose from the very complicated way in which the resultant forces behave. Fig. 3.4, for example, shows the variation, hour by hour, of the tractive forces at a particular place, and diagrams of this sort for tidal streams are not uncommon. The double loop on the curve is due to the existence of the diurnal and semidiurnal components.

Diagrams representing the variations of actual tidal streams yield many complicated curves of the general type of Fig. 3.4, but, as in the case of the tractive force, the diurnal and semidiurnal components taken separately yield elliptical curves.

21.3. Rotatory tidal streams due to gyroscopic forces

We shall now consider the effects of the gyroscopic forces on a tidal stream. For this investigation we shall not trouble ourselves as to how the tidal stream has been generated, but we shall suppose that in the absence of gyration a tidal stream would be flowing rectilinearly backwards and forwards, so that we do not complicate the investigation by trying to consider together the components of a stream already rotatory from other causes. In other words, we shall consider for simplicity a stream flowing, say, alternately east and west, and it is understood that exactly similar arguments can be applied to a stream flowing north and south, and therefore to any stream having components to the east and north. In the following investigation we shall consider streams and tides in the northern hemisphere, and it must be understood that for streams in the southern hemisphere the rotations will be reversed.

The gyroscopic forces, as we saw in Chapter XX, produce changes in velocity and the magnitudes of the forces are directly proportional to the velocity of the stream. For precision, let a solar semidiurnal stream flowing east or west in the absence of gyroscopic forces be called the primary stream, and let it be flowing east from hour 0 to hour 6, so that its maximum velocity occurs at hour 3. In the northern hemisphere, therefore, the gyroscopic force will be directed to the right of the primary motion, that is, to the south from hours 0 to 6, and it will be greatest in magnitude when the eastward velocity is greatest, at hour 3. Thus the resultant of the gyroscopic force is to give a component of velocity to the south, and when the rhythm is fully established, the stream will have a primary component of velocity to the east and a secondary component of velocity to the south.

Now the immediate effect of a force is to give an acceleration, a change in velocity, and velocity is not momentarily changing when it is at a maximum or

minimum, but it is changing most quickly, or the acceleration is greatest, at times half-way between the times of maximum and minimum velocities. Thus the gyroscopic force, and therefore the acceleration, to the south is greatest when the primary velocity to the east is greatest, at hour 3. Hence from hour 3 onwards the south component of velocity must continuously increase and it will reach a maximum at hour 6 when the acceleration southwards is zero (because the gyroscopic force is zero, since the primary velocity to the east is then zero).

Hence we see that the secondary component to the south generated by the gyroscopic force on the primary east component of a stream reaches a maximum three hours later than the primary stream. In the case considered, at hour 3 the stream is flowing east and has no south component, and at hour 6 it is flowing south and has no east component and so on. The effect of the gyroscopic forces is thus to cause streams to rotate in the clockwise direction in the northern hemisphere. In general, the secondary stream will have its maximum velocity to the right of the primary stream a quarter-period after the maximum of the primary stream.

Similar conclusions can be drawn for diurnal streams, and for any primary direction of stream. To be exact, however, it is necessary to consider the effects of the gyroscopic forces also upon the components of stream which are themselves directly caused by gyration, but the discussion then becomes very complicated without having recourse to mathematical investigation.

21.4. Rotatory streams from combinations of oscillations

The simplest possible representation of a tidal stream at any one place arises when the components at right angles can be expressed as

$$A \cos nt \quad \text{and} \quad B \sin nt$$

respectively. The possibility of expressing tidal stream constituents in this form is examined in Art. 21.7, where it is shown that this is always possible and involves a special time origin, with two components in the special directions of the major and minor axes of the ellipse as in Fig. 21.1.

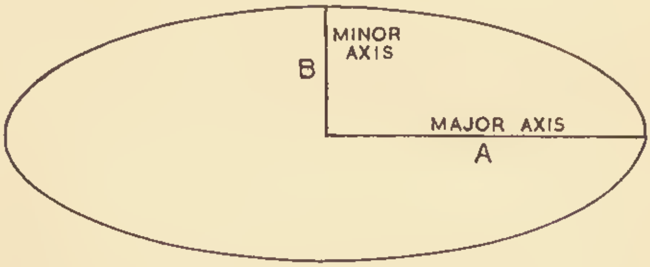

Fig. 21.1. Tidal stream ellipse.

If the component $A \cos nt$ is in the east direction and is the primary oscillation of the preceding article then the value of the secondary oscillation due to the gyroscopic forces is $B \sin nt$, provided that B is negative, for we showed that the southward component must be a maximum in the northern hemisphere a quarter-period after the maximum of the east-going stream.

Now the conditions requisite for two such components are very varied, but the simplest and most general oscillations giving these relations are two standing oscillations at right angles, one being a quarter-period behind the other. In a simple basin such as a rectangle the occurrence of two standing oscillations at right angles will produce rotatory streams unless the two oscillations are synchronous. However small the time-difference between the two oscillations, there will be rotatory streams, but with small time-differences the stream diagrams are very elongated ellipses. The rotation can be either in the clockwise or anti-clockwise direction according to the time-differences between the oscillations.

Another obvious possibility is that of a progressive wave along a coast and a standing oscillation at right angles to the coast. The conditions are readily seen to

be in their simplest form when at a given place the progressive wave makes high water synchronously with the standing oscillation, for at high water the streams due to the progressive wave are at their maximum in the direction of travel of the wave (Art. 17.2) while the streams from the standing oscillation are zero (Art. 18.1). Rotation can be in the clockwise or anti-clockwise direction according to whether the progressive wave travels one way or the other at right angles to the direction of the standing oscillation and also according to the time-differences between the oscillation, and of course very varied ellipses will result according to the magnitude of the time-difference between the respective high waters of the two oscillations.

21.5. Tides and tidal streams in channels of variable depth

A very important case of tidal motion occurs when the walls of a channel are not vertical (as was supposed in Chapters XVII–XIX), and we shall briefly discuss the subject of the elevation of tide across a channel in which the depth is not constant.

There is a wide-spread opinion to the effect that a progressive wave will move more slowly at the sides of a channel than in deeper water, based on the fact that in water of uniform depth h a wave travels at a rate equal to $\sqrt{gh}$, so that the rate of travel increases with the depth. According to this idea, high water will be earlier in the middle of the channel than it is at the sides (see Fig. 24.2). To show the fallacy of this, suppose we take a simple case of two channels side by side with a common partition, and such that one is deeper than the other. If at a certain moment the two waves are at high water together, then after the lapse of a certain time one will be in advance of the other, and ultimately one will have its high water at a place where the other is giving low water. Now according to the idea quoted above, it should make no difference if the partition is removed, whereas commonsense reasoning shows us that there would be a violent rush of water across the channel in order to equalise the elevations of water across the channel.

In nature this effort to keep the elevations as near to equality as possible would be effected more gradually, and one result would be that the wave as a whole would move at a rate corresponding to some value of a mean depth (across the channel, not necessarily the arithmetic mean). A little variation in the time of high water across the channel might occur, but any serious differences would tend to be smoothed out by calling into action streams transverse to the channel. In fact, the very rapid equalisation of levels in the artificial case just discussed would be effccted by streams transverse to the channel and in nature this would be done more gradually by the same agency.

Thus we conclude :—

(*a*) that it is a fallacy to suppose that it is high water appreciably earlier in the deeper water at the centre of a channel than it is in the shallow water at the sides ;

(*b*) that whereas in a channel of uniform depth the streams appropriate to a progressive wave flow to and fro in the direction of the channel, in a channel with shelving sides there are also streams across the channel, so yielding rotatory streams.

21.6. Tidal streams near shelving coasts

It is clear that at high water there is no necessity for transverse streams to be flowing, as there is no increase of elevation taking place anywhere, at least not to any great extent, across the channel, and it is only as the beaches are being uncovered that the water will begin to acquire a movement towards the centre. Hence the transverse streams will have zero rates at both high and low water, and their maximum rates will thus be at half-tide. But in a progressive wave in a channel of uniform depth, the streams have their maximum rates at high and low water (Art. 17.2), and the direction of flow at high water is in the same direction as the direction of

propagation of the wave. This will still be the case on the whole, but in addition there will be the *compensatory transverse streams*, if we may so call them.

Suppose the channel is in the east and west direction and that a progressive wave is travelling east, with high water at a certain place at hour 0, and low water at hour 6, if we suppose the tide to be semidiurnal in character. Then Fig. 21.2a shows the streams at hours 0 and 6, the streams at hours 3 and 9 having zero rates, if the streams due only to the progressive wave are concerned. The compensatory transverse streams are shown, however, in Fig. 21.2b at hours 3 and 9, and there are zero values at hours 0 and 6. Both figures give directions of streams at points near each coast. At hour 3 the tide is falling so that the compensatory streams are directed away from the coasts. These two diagrams are combined in Fig. 21.2c, for the chosen hours. For intermediate hours the streams would be intermediate in direction and in velocity.

This diagram is very instructive, for it shows that on the coast to the left of the direction of travel of the progressive wave (the north coast in this example) the

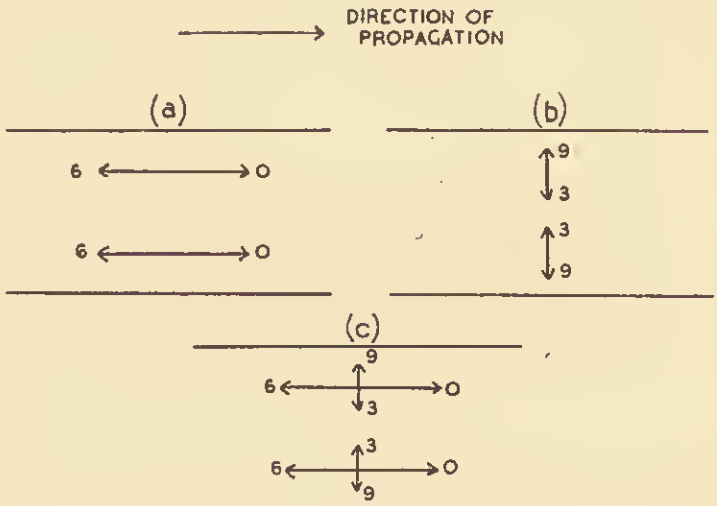

FIG. 21.2. Tidal streams near shelving coasts.

streams rotate in the clockwise direction, while near the right bank the streams rotate in an anti-clockwise direction.

A long shoal or bank in the middle of a wide channel would in the same manner give transverse streams in opposite directions on the two sides of the shoal.

Similar conclusions can also be drawn for diurnal oscillations or oscillations of any period, but all these conclusions are subject to modification when counteracting forces exist. All the contributory causes of rotatory tidal streams may act together or in opposition to one another, and in addition, there is the counteracting force arising from the gradients of elevation set up by streams flowing towards coasts. Thus the actual tidal conditions will involve many considerations if a detailed explanation is desired for the streams in any locality.

*21.7. Elliptical representation of tidal streams

It is sometimes desired to express tidal streams in the elliptical form with the components of stream taken along the axes of the ellipses.

Let the east and north components of streams be denoted by

$$\left.\begin{array}{l} A_1 \cos nt + B_1 \sin nt \\ \text{and} \quad A_2 \cos nt + B_2 \sin nt \end{array}\right\} \quad . \quad . \quad . \quad . \quad (21.7a)$$

where n is the speed of the motion, and t is taken from any assigned time origin.

* See par. 1, page vii.

Then the square of the velocity of stream at any moment is given by

$$W^2 = (A_1 \cos nt + B_1 \sin nt)^2 + (A_2 \cos nt + B_2 \sin nt)^2$$
$$= C \cos^2 nt + D \sin^2 nt + 2E \sin nt \cos nt \quad . \quad . \quad . \quad (21.7b)$$

where

$$\left.\begin{aligned} C &= A^2{}_1 + A^2{}_2 \\ D &= B^2{}_1 + B^2{}_2 \\ E &= A_1B_1 + A_2B_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (21.7c)$$

This can be written as

$$W^2 = F \cos 2nt + E \sin 2nt + G \quad . \quad . \quad . \quad (21.7d)$$

with

$$\left.\begin{aligned} F &= \tfrac{1}{2}(C - D) \\ G &= \tfrac{1}{2}(C + D) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (21.7e)$$

and this again as

$$W^2 = G + H \cos 2(nt - a) \quad . \quad . \quad . \quad . \quad (21.7f)$$

where

$$\left.\begin{aligned} H \cos 2a &= F \\ H \sin 2a &= E \\ H^2 &= E^2 + F^2 \\ \tan 2a &= E/F \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (21.7g)$$

Clearly, the velocity has a maximum value

$$\left.\begin{aligned} W_1 &= \sqrt{G + H} \quad \text{when } nt = a \\ W_2 &= \sqrt{G - H} \quad \text{when } nt = a + 90^\circ \end{aligned}\right\} \quad . \quad . \quad (21.7h)$$

and a minimum value (the second line above).

If θ_1 and θ_2 are the directions in an anti-clockwise direction from E when the velocities are W_1 and W_2 we have from (21.7a) and (21.7h)

$$\left.\begin{aligned} W_1 \cos \theta_1 &= A_1 \cos a + B_1 \sin a \\ W_1 \sin \theta_1 &= A_2 \cos a + B_2 \sin a \end{aligned}\right\} \quad . \quad . \quad . \quad (21.7i)$$

$$\left.\begin{aligned} W_2 \cos \theta_2 &= A_1 \cos (a + 90^\circ) + B_1 \sin (a + 90^\circ) \\ &= - A_1 \sin a + B_1 \cos a \\ W_2 \sin \theta_2 &= - A_2 \sin a + B_2 \cos a \end{aligned}\right\} \quad . \quad (21.7j)$$

From these expressions values of W_1 and W_2 agreeable with (21.7h) can be determined, together with the corresponding values of θ_1 and θ_2. It will be found, and can be proved, that we have either

$$\theta_2 = \theta_1 - 90^\circ$$

corresponding to clockwise rotation, or

$$\theta_2 = \theta_1 + 90^\circ$$

corresponding to anti-clockwise rotation.

CHAPTER XXII

KELVIN WAVES AND AMPHIDROMIC SYSTEMS

22.1. Kelvin waves

THE existence of boundaries or coasts to channels or seas profoundly alters the character of tidal motion, as we have already partly seen, and in this chapter we shall consider some of the more remarkable phenomena associated with the effects of boundaries.

In narrow channels, the streams transverse to the channel must necessarily be small, but the gyroscopic forces due to movement along the channel are still operative. If these forces cannot produce streams they must be counteracted by other forces, and reaction is provided by the hydraulic forces due to the tilting of the surface of the water. We thus see the possibility of a wave which has not the same elevation across the channel. Waves of this type were first investigated by Sir William Thomson (afterwards Lord Kelvin) and they are usually called *Kelvin Waves*.

As an example, take the case of a progressive wave, and suppose for simplicity it is semidiurnal in character. At hour 0 at a certain place, let it be high water. Then the stream in the absence of gyration is in the direction of progression of the wave, say to the east, and at hour 0 it has maximum value, as was shown in Art. 17.2. In the absence of boundary walls to the channel, the gyroscopic forces would tend to bring about a transverse stream to the south reaching a maximum 3 hours later. In a narrow channel, however, transverse streams of the magnitudes involved cannot occur. The gyroscopic forces must therefore be counter-acted by the hydraulic forces set up by a gradient of the surface, positive to the south, so that the elevation is greater on the right than on the left. This gradient is a maximum when the gyroscopic forces are greatest, that is, when the primary velocity is greatest, at high water.

Hence the gyroscopic effects are such as to magnify the range of tide on the right bank of the channel (in the northern hemisphere).

These results may be derived also from considerations of steady motion. If we suppose ourselves to be travelling with the progressive wave at any particular point of the profile, then we shall not experience any change in the conditions. The elevation will appear to be fixed at the point of observation while the stream will appear to flow backwards at an invariable rate. If gyration gave a velocity to the right of the stream, then this velocity would also be invariable, but this is impossible as the gradient would increase to the right throughout the motion of the wave. Obviously a limit must be reached when there is set up the appropriate gradient, after which conditions would remain unchanged.

Thus in the case of a narrow channel a progressive wave could exist with no transverse streams but with a greater range of tide on the right of the channel than on the left.

There can be all states between the two extremes of

(*a*) no transverse gradients, with rotatory streams, and
(*b*) a transverse gradient, with no transverse streams.

There are so many factors to be taken into consideration in any given case, that it is not easy to lay down general laws.

22.2. Amphidromic systems due to gyration

A very interesting and important system of tidal movement occurs when standing oscillations are modified by gyroscopic forces. We have already discussed

in Chapter XXI the effects in the streams when all counteracting forces are ignored, and in the article just considered we have indicated a special type of wave associated with movement along a coast. We shall now proceed to discuss the effects of gyroscopic forces where a standing oscillation occurs across a sea. For simplicity, and in pursuance of our policy of considering parts of the phenomena, we shall not consider how the primary oscillation is set in being or how it is maintained, and we shall consider the oscillations in a simple shape of basin only, as in Chapter XVIII.

In the case of a rectangular basin (Art. 18.1) we saw that when there is high water at one end and low water at the other, the velocity everywhere is zero, but that when the surface is level the velocities have their maximum values in the direction where the elevation is increasing. When the gyroscopic forces are operative

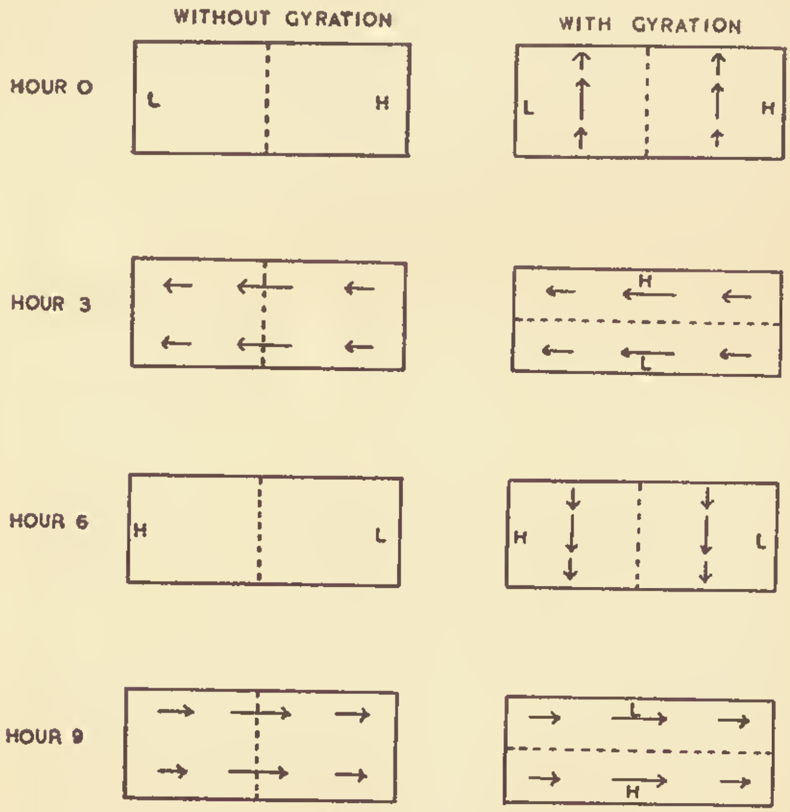

FIG. 22.1. Effects of gyration on standing oscillation in rectangular basin. (The components of the streams are drawn at their maximum rates ; the broken lines join points at mean level.)

they will build up a surface gradient to the right of the path of the stream so that there will be a subsidiary elevation of the water to the right. This gradient will increase with the velocity, and will be greatest therefore on the centre line, where the nodal line existed in the case of non-gyrational motion. For simplicity let us take a semidiurnal west and east oscillation along a rectangular basin in the northern hemisphere as in Fig. 22.1. At hours 0, 3, 6 and 9 we have the successive phases shown on the left of the diagram appropriate to the conditions when there is no gyration. The broken line is the nodal line. At hour 0 when it is high water (H) to the east of the nodal line and low water (L) to the west of it there are no streams. Three hours later the surface is level and the streams are at their maximum strength, tending to produce high water to the west at hour 6. Now when gyration occurs its effects at hour 3 are to give a gradient upward to the right of the stream line so that the surface is at high water at the north end of the original nodal line and at low water

to the south of it, these conditions being reversed at hour 9. But these north and south oscillations require appropriate streams so that at hour 0 there must be a stream flowing to the north to produce the state of elevation three hours later.

The result now is that high water occurs at hours 0, 3, 6, 9 at the centres of the sides of the rectangles (see Fig. 22.2) in anti-clockwise rotation. The original nodal line has shrunk to a central point at which no tidal changes occur, and such a point is called an *amphidromic point* because the *cotidal lines* (that is, the lines joining points at which high water occurs at the same time) radiate from this point and rotate round it. The range of tide in such a system increases with distance from the amphidromic point.

Hence, we see that in the northern hemisphere a standing oscillation will become so modified by the effects of the gyration of the earth that nodal lines will degenerate into points, round which the tidal oscillations appear to rotate in an anti-clockwise direction. In the southern hemisphere the rotation of the amphidromic system is in a clockwise direction. In both cases the period of rotation in the example is 12 hours, but in general it will be the tidal period, approximately 12 hours for semidiurnal tides and 24 hours for diurnal tides.

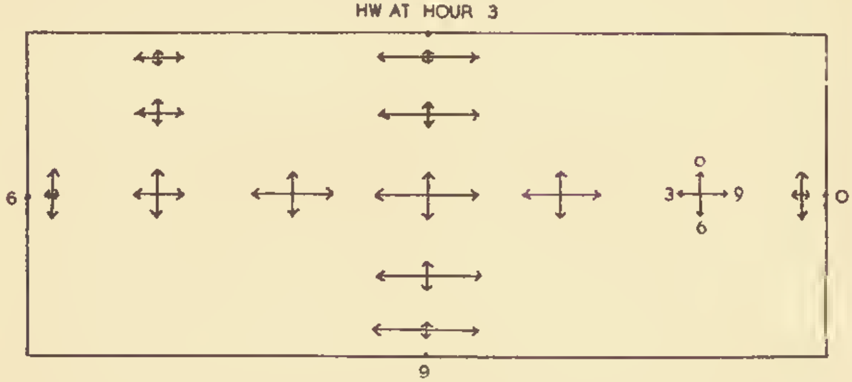

FIG. 22.2. Semidiurnal tides and tidal streams in revolving rectangular basin.

The rotation of the streams is of interest and Fig. 22.2 shows the magnitudes and directions of the streams at a few selected places for the hours 0, 3, 6, 9. To avoid confusion we shall note first that for all places the streams run northward at hour 0, westward at hour 3, southward at hour 6, and eastward at hour 9 ; at intermediate hours, though the directions of the streams at all places are between the same two cardinal points, as for instance from hour 0 to hour 3 all directions are between north and west, the actual direction at any place depends on the relative magnitudes of the east and north components at the hour and place. For any given *place* the east component is maximum at hour 9, and for any given *time* the east components are maximum at the centre of the sea, and are very small near the bounding sides. Thus at hour 9, the centre of the sea has a very large maximum east component, whereas nearer the coast there will be a small maximum east component. Then along the centre line running north and south, in the example, the east components of stream have all the same strength, whereas the north component varies from maximum at the centre to zero at the boundaries. Along the central line running east and west, in the example, the values of the north components of stream are the same at all points whereas the east components are at maximum strength in the centre. The rotations of the streams are in the same direction, namely anti-clockwise, in all cases. In the southern hemisphere rotation is in a clockwise direction.

We immediately note that on the boundaries of the sea the streams run practically to and fro, and that the maximum strengths of streams in the same direction as the direction of rotation of the amphidromic system occur at the times when it is local high water on the coast. But we showed in Art. 17.2 that a progressive wave

is characterised by having maximum streams in the direction of propagation at the time of high water.

Thus, if we confine our attention to the coasts of the sea, we have an apparent phenomenon of a progressive wave round the sea. This is a very curious result, but it is clear that it is a special type of wave. Hence the result of gyration on a standing oscillation, through producing another standing oscillation at right angles to it and differing in phase by a quarter-period, has been to produce a progressive wave of amphidromic type. It may be noted that the amplitude of this wave diminishes away from the coast, so that we have this wave somewhat like the Kelvin wave in this respect though the complete system for the whole basin cannot be described by a single Kelvin wave.

22.3. Amphidromic systems in small deep seas

We are led to enquire whether amphidromic systems are entirely due to the gyration of the earth, and we proceed to investigate the tides in a small deep sea.

If a sea is deep, then the transport of water required in the tidal oscillations is effected over a very large vertical area, and thus the streams are small. Consequently the gyroscopic forces, which are proportional to the velocity of the streams, can be ignored. The system discussed in this article is therefore quite different from that discussed in the previous article.

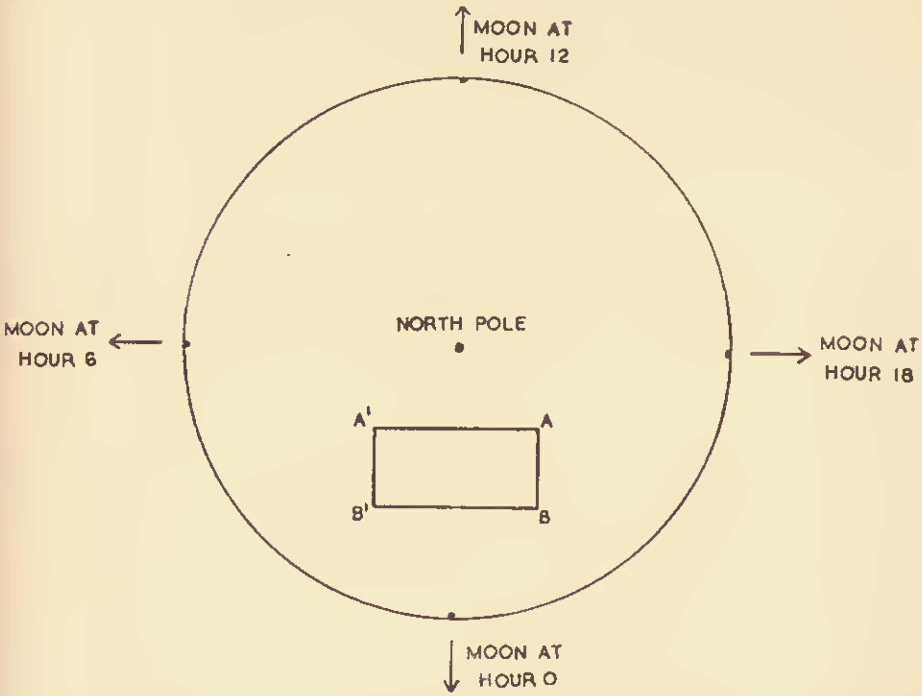

FIG. 22.3. Slope of surface in small seas, directly due to lunar semidiurnal tide-generating forces.

If the sea is of small dimensions, the transportation of water is quickly effected, and in consequence the water responds very rapidly to the direct effects of the tide generating forces. The equilibrium conditions are easily reached and the elevation of the surface can be deduced like that of the equilibrium tide. Thus, whereas it is impossible for equilibrium conditions to be reached in the case of large oceans, in the case of a small deep sea the conditions are more favourable. The surface then takes a position at right angles to a pendulum hanging under the ordinary gravitational forces of the earth, and under the deflecting power of the tractive forces, as was discussed in Art. 4.1.

We need only consider the lunar semidiurnal tide, and for simplicity of reasoning we shall assume a rectangular shape of sea. Let the sea be denoted by ABB′A′ where AB, A′B′ are running north and south. In Fig. 22.3 suppose the earth to be seen from the pole of the heavens and to be relatively at rest, so that the moon appears to revolve round the earth. Then at hour 0, with the moon south of the sea, the tide-generating forces will give an equilibrium form with a gradient increasing upwards in the southerly direction. If the surface responds to this gradient (as it should do under the conditions) it can only be effected by the water along AA′ being depressed below the mean level while that on BB′ is elevated above the mean level. This is a necessary consequence of the invariability of volume. The surface is therefore parallel to the surface of the equilibrium tide that might exist under hypothetical conditions with water covering the whole earth.

The highest point will be at the centre of B′B, and the lowest point will be at the centre of AA′, at hour 0.

An hour later, as the moon moves westward the place of high water has moved westward also, and the surface still forms a plane which, so to speak, is pivoted at its central point. At hour 6, we must have reached the stage of low water on BB′, and so on. The resulting conditions are set out in Fig. 22.4. The gradient is everywhere uniform at any instant so that the range of tide increases with the distance

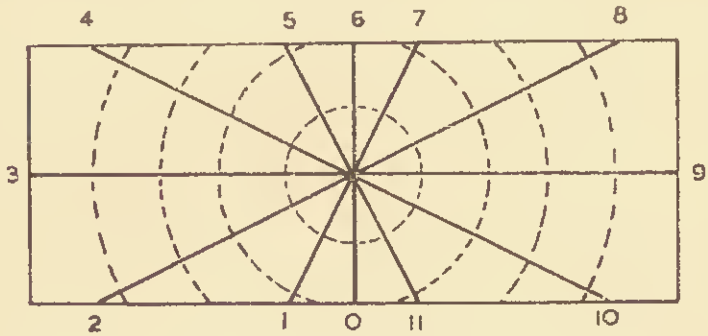

Fig. 22.4. Amphidromic system in small seas, directly due to lunar semidiurnal tide-generating forces.

from the centre, at which there is zero range. This point of zero range is called an amphidromic point. Lines joining points having equal ranges of tide are called *co-range lines*, and lines joining points at which high water is simultaneous are called *cotidal lines*.

In this particular case of a small deep sea (but not universally) the cotidal lines are straight lines and the co-range lines are circles or parts of circles.

The angles between the cotidal lines depend upon the latitude of the centre of the sea. At the pole, with zero declination of the moon, the lines will be equally spaced but the tide range will everywhere be infinitesimal. If the sea is on the equator, then the gradient at zero hour will be zero, since the tide-generating forces are zero. Hence the oscillation will become a pure standing oscillation and the water will oscillate about the meridional centre line of the sea, in which case we have the normal standing oscillation with high water everywhere on one side of the nodal line, and low water everywhere on the other side.

In between these two extremes, we have the normal cases where the angle between the cotidal lines will depend upon the latitude. They will be like a closed fan near the equator.

This case finds exemplification in enclosed seas like the Caspian sea, where the ranges of tide and the time of high water, apart from local effects of configuration, are practically the values given by the equilibrium relations as explained above.

Since the gradient of elevation is that of the equilibrium tide, then it must be anticipated that the range of tide in such a sea must be much smaller than the

equilibrium tide over the whole earth, for the elevation obviously depends not only upon the gradient but also upon the distance through which the gradient is experienced. As the lunar equilibrium tide has a maximum semidiurnal semi-range of 0·88 ft., the tides in enclosed seas are therefore very small indeed.

The diurnal systems arising from tide-generating forces may have clockwise or anti-clockwise rotation according to latitude.

The above investigation, of course, has not been made on account of the importance of the tides in small seas, but on account of the important principle that amphidromic systems can result from the direct effects of the tide-generating forces, and we choose the small sea because it is by no means easy to deal satisfactorily with large oceans.

22.4. General remarks on amphidromic systems

The foregoing investigations deal with amphidromic systems which are generated in different ways. In one instance (Art. 22.2) all external forces are ignored, for we deliberately refrained from considering how the primary tide was maintained, and in the other (Art. 22.3) the streams are so small that gyroscopic forces are entirely negligible. In both cases, we took semidiurnal tides for the purpose of description, but the principles can be readily applied to any species of tide. Also, rectangular seas were taken for simplicity, but the principles can be applied to any seas, and in the case of the gyroscopic effects there is no restriction of size or shape of sea or ocean, nor is there necessity to consider a closed sea, for the principles can be applied to gulfs. The amphidromic system deduced in Art. 22.3, however, is only applicable to small deep enclosed seas, but the shape is immaterial. There are other ways in which the two amphidromic systems may be considered, for in the cases illustrated the amphidromic rotations are in opposite directions.

In general, however, these investigations show that two important causes are in operation to produce amphidromic systems and in large oceans these causes are to be considered together and not separately. Thus, the problem of determining theoretically the tides in the large oceans is exceedingly complicated, but the investigations made in this chapter serve to show that amphidromic systems of tides may he regarded as the rule rather than the exception.

CHAPTER XXIII

TIDES IN OCEANS

23.1. The approach to the ultimate state of knowledge

THE ultimate state of knowledge of the tides will exist when the tides can be explained quantitatively without reference to observation. Such an ideal is still very far from attainment, though steady progress is being definitely made towards it. We have seen that so far as the forces which give rise to tides are concerned there is no difficulty in stating these with as much accuracy as can be desired. The difficulties to be encountered are found in association with the physical properties of the fluid media on which the forces act, and the configurations of the basins of the oceans or seas.

The next stage of the mathematical process is in connection with the formulation of equations which represent

(*a*) the instantaneous relations between the movement of a small element of fluid and the forces acting upon it ;

(*b*) the important fact that in any given space not enclosing the free surface, large or small, the total amount of fluid is not altered, for the flow inwards must equal the flow outwards.

We have encountered simple cases of equations such as these in the investigation of waves. Bernoulli's equation is related to the general equations of motion, and the method used to express the equality of flow of water across sections of a channel is a special case of the general *Equation of Continuity*, as it is called. Now the general equations can be formulated with great exactitude, and many eminent mathematicians have critically scrutinised them, so that we can almost say that this stage of progress is complete.

Even so, the exact equations are seldom used by mathematicians because of the great mathematical difficulties involved ; the subject has developed by easy stages, involving the assumption of conditions which are not always even rough approximations to those actually existing. Fortunately, the equations of motion can be simplified very considerably, especially if the vertical motion is known to be small compared with the depth ; this is always the case except in estuaries and shallow seas, and even in these apparent exceptions the simplified equations are quite good approximations.

We have seen something of the character of these simplifications in connection with progressive waves, where we first assumed that the wave-length was large compared with the depth so that vertical oscillations could be ignored, compared with horizontal oscillations. Then we assumed the range of oscillation to be small compared with the depth, so that we could ignore effects which depend upon the squares and products of small quantities.

By so doing we obtained the law of propagation of a wave as a whole. On again examining the problems some of these assumptions were removed and we got a little closer to actuality.

Hence the complete solution to the problems of tidal motion is being sought by progressive stages.

23.2. The equilibrium theory

Though a great deal has already been said in connection with the equilibrium theory, it is perhaps desirable, in the historical survey that we are now making, to recapitulate reasons why the equilibrium theory formulated by Newton three centuries ago proved unable to account for tides as they exist upon the earth. It was

shown that the equilibrium tide, as it is generally understood, could only exist on an earth covered by water, and that its name implies a steady state resulting in the tidal protuberances in the water being in the line of centres of earth and moon, in the case of the lunar part of the tide. As the earth revolved it was supposed that it was possible for the water to move sufficiently rapidly for it to maintain this position. The failure of the equilibrium theory is thus due to the inertia of the water being too great for the necessary rapidity of movement. If the water were very deep indeed then the necessary transport of water would take place over a very big vertical area and so we see that the deeper the water the more nearly do we approach the conditions required for the equilibrium tide to be considered as practicable. The depths involved, however, are far beyond those experienced upon the earth.

23.3. Laplace's solution for an ocean covering the whole earth

It is very remarkable that as far back as the year 1776, the French mathematician Laplace gave a solution of the problem of tides in an ocean of constant depth covering the whole earth. This solution took full account of the inertia of the water and of the gyration of the earth, but it utilised mathematical methods that were by no means simple to understand. In fact the solution was disputed by eminent mathematicians but was finally vindicated in a remarkable paper in 1875 by Lord Kelvin, though his explanation, to many, only made the matter more mysterious than it really is. It is only in recent years that this theory can be said to have been simply explained, by more elementary methods than those used by Kelvin.

It is evident that it is impossible for us in this Manual to enter into the theory, but it is well to know that even such a simple ocean as that postulated by Laplace, on a rotating earth, is by no means easy to deal with. Some of Laplace's results are of very great interest. Before giving the actual results we can consider a simple deduction from the fact that water is supposed to cover the whole earth. If we suppose, for simplicity, that the moon is on the equator, then the ranges of tide at all points on the same latitude must be equal, for there can be no bias in favour of any one point. Again, there is no terrestrial reason why there should not be symmetry of elevation with regard to the moon. It would be natural to consider high water as occurring simultaneously at all points on the meridian plane through the moon, as occurs in the equilibrium tide, but actually we cannot go so far as that. We realise that on either side of the meridian there is a symmetry of tractive forces so that we cannot have the tide greater on one side than on the other. According to this argument, therefore, at points along the meridian, we may have high water or we may have low water satisfying the condition of symmetry, and this is what Laplace actually found.

Laplace's solution has been evaluated for four depths, 7,260, 14,520, 29,040 and 58,080 ft., corresponding to certain simple values of the constants involved in the solution, and the following table gives the relationship to the corresponding equilibrium expressions, for the semidiurnal tide, at the equator :—

TABLE 23.1

Laplace's solution for Ocean covering Whole Earth

Depth in feet	Depth in fathoms	Ratio of elevation at equator to equilibrium elevation
7,260	1210	— 7·4
14,520	2420	— 1·8
29,040	4840	11·3
58,080	9680	1·9

For depths much greater than 58,080 ft. the ratio steadily approaches unity, in accordance with the general theory as described above, and the equilibrium tide would be realised in such an ocean if the depths were very great. But, as the depth diminishes, the tide is progressively larger than the equilibrium tide until at some depth just less than 29,040 ft. the tide becomes infinitely large and for smaller depths

still the ratio is negative. This means that we then have, under the moon, low water and not high water. The tide is then said to be *inverted* whereas for large depths it is *direct*.

We shall not discuss the very interesting results obtained by Laplace for the diurnal and long-period tides. The phenomenon of *inversion* in which we get low water where we would expect high water from the character of the forces is of great interest and requires further explanation.

23.4. Forced waves in canals encircling the earth

While Laplace's theory is interesting, the ocean for which it applies is a long way removed from reality. We have seen in previous chapters that bounding land masses profoundly affect the movements of waves and we have also seen that in a small enclosed sea the character of the tidal motion is very different from that of the equilibrium theory. In the latter theory the tide might be considered as a progressive wave whose range depends upon latitude, while in the case of the small sea the tide is of the amphidromic type.

In 1845, Airy developed the theory of tides in relatively narrow canals, in connection with progressive and standing oscillations. The wave theory is undoubtedly an attractive one in many respects, for the conception of a wave is acceptable to the understanding of the average man, but unfortunately many amateur theorists have rushed in where Airy feared to tread. Like Laplace's work, Airy's work should be regarded as one of the stages in the scaffolding which helps to support the structure in the process of building. It reveals some truth but not all the truth.

There exists in nature a crude semblance of a canal in the waters of the southern hemisphere, north of the Antarctic continent. Here there is a wide belt of water encircling the earth, which might be called a re-entrant canal, and in which a progressive wave might be considered to move for ever without hindrance.

Let us consider, therefore, a canal running round the earth, between two parallels of latitude, and suppose that under these circumstances we can temporarily ignore the gyroscopic effects. Owing to the similarity of circumstances of each point on the same parallel of latitude, relative to the tractive forces, it follows that all points on it will have the same range of tide, and that all will have the same interval between the transit of the moon and the time of high water. Considerations of symmetry, exactly as in the case of Laplace's problem, indicate that we must have either high water or low water under the moon. This argument is applicable to all depths, and in the case of very large depths it covers the equilibrium expressions, with high water under the moon.

We thus have a progressive wave which is *forced* by the external tide-generating forces, and which does not travel like a *free* wave (that is, one moving under the influence only of the earth's gravitational force) with a speed equal to $\sqrt{gh}$, but at such a rate as will cause it to traverse the circumference of the circular canal in a period of a lunar day.

Let the rate of progression of this wave over the earth's surface be denoted by c so that c is the actual rate of the forced motion and not, in this article, the rate $\sqrt{gh}$. We then have *steady* conditions and if we suppose ourselves to be travelling with the wave at a rate c we can apply the complete expression for Bernoulli's equation (17 9r) :—

$$\frac{p}{\rho} + \frac{1}{2}(u - c)^2 + gH + P = \text{a constant} \qquad (23.4a)$$

where P is the potential of the tide-generating forces. Since these forces have a period of 12 lunar hours for the semidiurnal tide, so also will P, and we are not concerned with its actual value. At the surface the pressure (p) is equal to the atmospheric pressure, which can be taken as constant, and H can he taken as y, the elevation of the surface above the mean. Hence

$$\tfrac{1}{2}(u - c)^2 + gy = \text{constant} - P \qquad (23.4b)$$

The arguments used are almost exactly the same as those used in Art. 17.10 in connection with the free waves, and similarly we get

$$u - c = -\frac{ch}{h+y} = -c(1 - y/h) \text{ approximately} \qquad (23.4c)$$

Hence

$$(u - c)^2 = c^2(1 - 2y/h) \text{ approximately} \qquad (23.4d)$$

if we can neglect the square of y/h.

Hence we get the result

$$\begin{aligned}(g - c^2/h)y &= \text{constant} - P - \tfrac{1}{2}c^2 \\ &= \text{constant} - P\end{aligned} \qquad (23.4e)$$

since c is a constant also.

The average value of y round the canal must be zero, since we have a complete oscillation about mean sea level, and the average value of the potential P must be zero also, for the average value of the forces round a parallel of latitude is obviously zero. Hence the constant in (23.4e) is zero, and therefore

$$y = -\frac{P}{g(1 - c^2/gh)} \qquad (23.4f)$$

Now P is an external force, and is the same for all depths, so that in the case of a very large depth we get the equilibrium elevation (Art. 23.2) which we shall denote by $\bar{y}$ so that

$$\bar{y} = -P/g \qquad (23.4g)$$

Then it follows from this that

$$y = \frac{\bar{y}}{1 - c^2/gh} = \frac{\text{the equilibrium value}}{(1 - c^2/gh)} \qquad (23.4h)$$

If the actual value of c^2 is less than gh, so that the rate of travel of the forced wave is less than that of the free wave, then the actual elevation has the same sign as the equilibrium elevation, but the amplitude of oscillation is greater: that is, the tide is *direct*. If c^2 is greater than gh, so that the forced wave travels faster than a free wave, then the tides are *inverted*, so that there is low water where the equilibrium tide would have high water. When $c^2 = gh$ there is resonance.

It is instructive to return to the equation

$$gy - c^2y/h = -P \qquad \text{(see 23.4e)}$$

The term gy is the variable part of the potential energy due to the earth's gravitational force, while the variable part of the kinetic energy is given by $-c^2y/h$. The latter term is simply due to the fact that the velocity falls as the elevation increases, the velocity, that is, relative to a plane moving with the profile. In that stage where the tides are direct the energy of the system is principally potential energy, whereas in the inverted system the kinetic energy predominates. In the critical condition, that of a wave travelling with the velocity $\sqrt{gh}$, the two variable parts are equal and opposite in sign, so that no external energy is needed to keep the wave in being.

23.5. Canal and progressive-wave theories

The practical applications of the preceding theory are readily deduced. If the canal is at the equator then the circumference is 21,625 nautical miles. The lunar day is 24·84 mean solar hours, and therefore the rate of travel of the wave is 871 miles per mean solar hour. The rate of travel of a free wave can only reach this value if the depth is 11,200 fathoms. This is far greater than the mean depth of the oceans and in fact the ratio c^2/gh when the mean depth is taken as 2000 fathoms is

5·6, so that the tides in such a canal would definitely be inverted, and only about one-fifth of the equilibrium value (see 23.4h).

But if the canal is at latitude 45° the circumference is reduced in the ratio $\cos 45° = 0{\cdot}707$, so that in this case the critical depth would be half that at the equator. In general, the critical depth will be equal to that at the equator, multiplied by the square of the cosine of the latitude. The tides would thus become direct near the pole.

The canal theory for a system of parallel canals would thus indicate inverted tides over the greater part of the earth, with direct tides near the pole, and with a critical latitude at which resonance would occur. If it were possible to conceive the barriers between the canals to be then removed, this resonance would not occur, for it depends upon a fine balance between the variations of the potential and kinetic energies, and only a little energy would need to be absorbed by neighbouring canals in order to destroy the tendency to resonance. The transition from inverted to direct tides would then take place at places of zero range, in a more natural way than that of resonance.

But it is clear that such a system of canals entirely ignores the gyroscopic effects associated with the gyration of the earth. The canal theory therefore does nothing to explain tides in general, but the above discussion may help to clarify the minds of would-be theorists, many of whom maintain that the belt of water surrounding the Antarctic continent is sufficient for the generation of tides, and that when such oscillations have been set up they generate free waves northward, (1) towards Alaska through the Pacific ocean, (2) towards Arabia through the Indian ocean, and (3) towards Iceland through the Atlantic ocean. It was thought at one time that some evidence for this was to be found in that the solar tides appeared to travel rather more slowly than the lunar tides, and the progressive retardation appeared to be explained by the progressive-wave theory. Many detailed observations have not supported this view, and there are many other serious discrepancies. A very important objection is that a simple free wave could not travel up these finite channels without suffering reflection and that any reflected wave would seriously affect the character of the motion. Reflection must take place unless the energy of the wave is somehow absorbed to generate heat and there is every evidence at the north pole that there is no accumulation there of the heat of dissipation.

Again, the width of the Atlantic ocean is quite large enough for resonance to occur within the ocean. A mean depth of three miles would only require a sea 3000 miles long to give resonance with a period of 12 hours. Hence we see that the tides in the Atlantic ocean are in all probability generated within the ocean and owe little, if anything, to the tides in the southern ocean.

The consensus of opinion among all mathematicians of recent years has been to the effect that the progressive-wave theory cannot be regarded in any way as a satisfactory explanation of the tides.

23.6. Standing-wave theory

One of the criticisms launched against the progressive-wave theory is that sooner or later land barriers will cause reflected waves, and these will so combine with the primary wave as to change the apparent character of the motion. It was shown in Art. 18.3 that in a canal the effect of reflection is to give a standing oscillation in which high water occurs simultaneously over the whole canal if it is short or over a large fraction of it if it exceeds a certain length.

It is impossihle to avoid the conclusion that any theory of tides must take account of these standing oscillations, and regard must be paid to the critical depths of the oceans. Both these factors are ignored in the "explanations" of the tides associated with the wave theories as expounded by people ignorant of the underlying principles which were clear to men such as Airy.

The importance of resonance was emphasised by Harris, a mathematician of the United States Coast and Geodetic Survey, at the end of the last century, and he attempted to select special portions of the water surface of the earth, in which the

free oscillations might have speeds nearly equal to that of a constituent of the tide-generating forces. On this basis he constructed a system of explanations of tide throughout the oceans.

Again, Harris assumed that each area could respond to the tide-generating forces without feeling any appreciable reaction from the oscillations in neighbouring areas, and that the gyroscopic effects could be ignored. We saw that in the case of parallel canals, where one was of critical depth, if the boundaries were removed the state of resonance would disappear, owing to interaction with the water in neighbouring canals. These two assumptions thus make it extremely doubtful whether Harris's hypothesis can lead to a satisfactory explanation of the tides in any of the oceans, but the importance which he attached to resonance and to standing oscillations has undoubtedly made for progress in the understanding of the problems of explaining actual tides.

The criticism relating to the interaction of neighbouring areas relates principally to the cases where an ocean was split up into areas which are not separated even partially by any land boundaries. An ocean such as the Atlantic ocean is largely land-locked, and thus a first approximation to the tidal conditions might be reasonably anticipated by considering the ocean as entirely closed.

23.7. Solutions for mathematically simple basins

Since small variations in the depth of the sea or ocean cannot have much influence on the general tidal régime it is a reasonable approximation to take the depth constant or to change slowly according to sure geometrical law. Small irregularities in the coast line can also be ignored in considering the general character of the oscillation in a given basin. Thus seas can be considered as rectangular, gulfs can be considered as rectangular with constant depth or with diminishing depth ; or the gulf may be considered as a converging gulf. We have dealt with some of these problems already, and for our present purpose we can reasonably assume that if the area of the gulf is small compared with that of the sea or ocean, its effect can be temporarily ignored.

Hence mathematicians have been led to consider the tides that might occur in geometrically simple basins on a revolving earth. The case of an ocean completely covering the earth was dealt with by Laplace, and until recent years no cases had been solved apart from those of small seas, narrow gulfs, and narrow canals of definite length, or encircling the earth, or extending from pole to pole. The main conclusions from these mathematical developments have been already dealt with.

Laplace's work on oceans covering the earth was extended by Hough in 1897 to include the effects of the mutual attraction of the water, but his results were of technical interest only.

Goldsbrough in 1913 dealt with the case of a polar ocean, and obtained results summarised as follows :—

TABLE 23.2

Polar Ocean (bounded at Latitude 60°). Ratio of Tide at Boundary to Equilibrium Tide

Depth in feet	Depth in fathoms	Diurnal tides	Semidiurnal tides
7,260	1210	− 2·86	1·07
14,520	2420	− 1·16	1·03
29,040	4840	− 0·19	1·01
58,080	9680	0·39	1·00

Thus the semidiurnal tide is practically the same as the equilibrium tide. The diurnal tides are interesting, because Laplace found that in his problem, with constant depth, there is no diurnal tide but there are diurnal streams. The effects of boundaries are thus made evident once again, because in the case of the polar ocean the diurnal tide may be larger than the diurnal equilibrium tide.

Goldsbrough also obtained in 1914 solutions for an ocean bounded by two parallels of latitude, and illustrated the solutions as follows :—

TABLE 23.3

Zonal Ocean bounded at Latitudes 30° and 14° 30′. Ratio of Tide to Equilibrium Tide

		Latitude 30°		Latitude 14° 30′	
Depth in feet	Depth in fathoms	Diurnal tides	Semidiurnal tides	Diurnal tides	Semidiurnal tides
7,260	1210	− 0·10	0·48	− 0·53	0·39
14,520	2420	− 0·39	− 0·58	− 0·99	− 0·07
29,040	4840	− 0·84	− 1·46	− 1·63	− 0·78
58,080	9680	− 42·6	1·81	− 31·0	0·37

TABLE 23.4

Zonal Ocean bounded at Latitudes 30° and − 14° 30′. Ratio of Tide to Equilibrium Tide

		Latitude 30°		Latitude − 14° 30′	
Depth in feet	Depth in fathoms	Diurnal tides	Semidiurnal tides	Diurnal tides	Semidiurnal tides
7,260	1210	1·95	− 0·59	3·48	− 0·46
14,520	2420	1·21	4·49	2·16	− 3·85
29,040	4840	0·45	− 1·11	2·19	− 0·70
58,080	9680	− 3·00	− 6·61	0·69	− 3·44

The effects of boundaries are again apparent, and it must be obvious that general reasoning of the popular type is not likely to be very satisfactory.

The problem of tides in an ocean bounded by two meridians is of very great interest and importance, for the results for meridians 60° apart might then throw light on the Atlantic tides, and those for 180° apart might be useful in connection with the tides of the Pacific ocean. The mathematical difficulties are extremely great, and the problem has been under continuous investigation by Proudman and Doodson for nearly 20 years with a view to providing solutions illustrating all widths of oceans and all depths. This ambitious programme is not yet complete, but at the present time the state of the work is as follows :—

Meridians 180° apart : the mathematical theory is complete and the diurnal and semidiurnal tides have been evaluated and illustrated.

Meridians up to 90° apart : a special semidiurnal case has been computed for all widths.

As a preliminary to this work, the tides on a non-rotating earth, in oceans bounded by meridians, were computed.

The results are of extraordinary variety, and the tidal régime is in many instances very complicated. Amphidromic systems predominate and there may be quite a number of such systems occurring together. Within the scope of these pages it is not possible to describe the whole of the results, and we shall confine ourselves to illustrating the tidal systems in four oceans, all of the same depth (14,520 ft.) and varying widths of 50°, 70°, 90°, and 180° (Figs. 23.1 to 23·4). The cotidal lines are drawn for the tidal constituent K_2 on account of certain simplifications in the mathematical formulæ for that constituent, but the systems for M_2 will be much the same. The numbers on the cotidal lines are the values of the phase-lags in degrees (g, referred to the central meridian). The co-range lines have not been included, for the sake of simplicity, but the maximum amplitudes of tide for the four cases are 7·9H, 4·9H, 6·3H, and 6·6H respectively, where H is here the maximum amplitude of the equilibrium tide. It is evident that the tides are much greater than the equilibrium tides and other results show that resonance will occur in an ocean bounded by meridians 78° apart, for the same depth (14,520 ft.).

These figures show that resonance may be expected in an ocean comparable in size and, roughly, in shape, with the Atlantic ocean. This was also shown by

investigations by Goldsbrough and Colborne for an ocean bounded by meridians 60° apart, for a depth comparable with the mean depth of the Atlantic ocean. All these results suggest that the Atlantic tides are generated within the Atlantic ocean.

It is necessary to utter a word of caution as to the use of the above investiga-

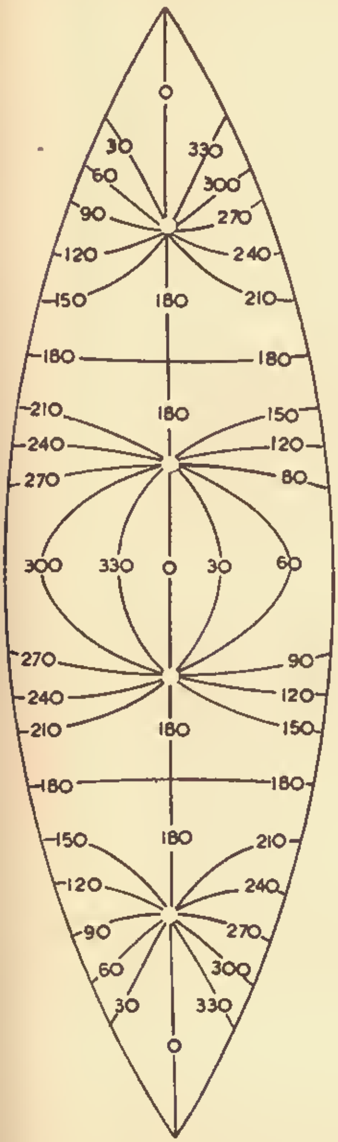

FIG. 23.1. Tides in ocean bounded by meridians 50° apart. (Depth 14,520 ft.)

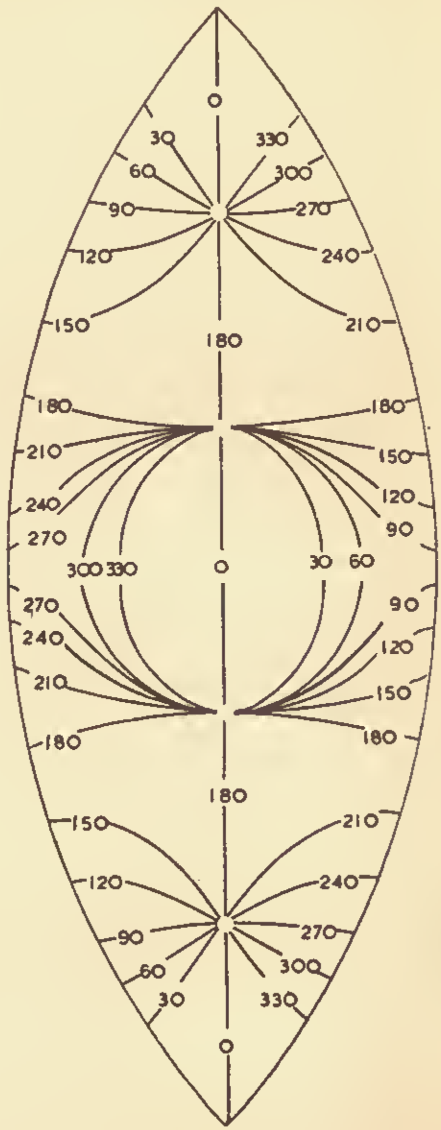

FIG. 23.2. Tides in ocean bounded by meridians 70° apart. (Depth 14,520 ft.)

tions. Actual oceans differ so greatly in shape from the geometrically simple cases submitted to investigation that it would be folly to expect to apply the above figures to the Atlantic ocean. The value of the investigations consists in being able to determine types of amphidromic systems, and to deduce approximate conditions for resonance to take place. The two systems of cotidal lines given in Figs. 23.2 and

23.3 have peculiarities which would have caused doubts to arise if they had been suggested for an actual ocean.

It is evident that while very considerable progress has been made in recent years towards the solution of the problems of the tides in actual oceans we are still very far from the ideal state in which the tides over the world, with its varied shapes of

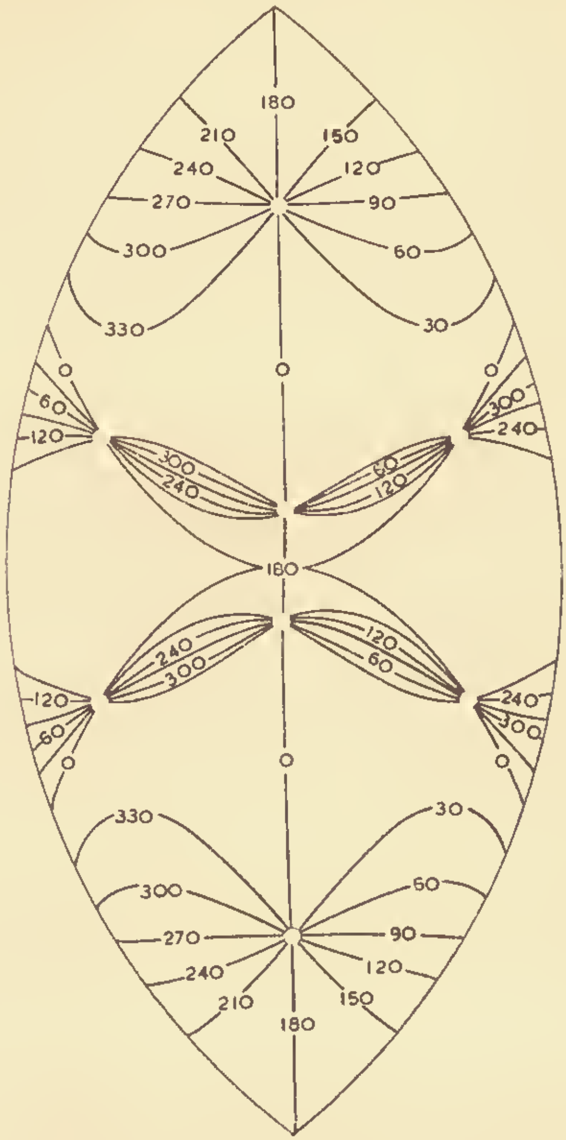

FIG. 23.3. Tides in ocean bounded by meridians 90° apart.
(Depth 14,520 ft.)

oceans, can be computed direct from the forces without recourse to observations, and indeed we are still unable to give with precision cotidal lines for any ocean. One of the most important considerations which will need to be taken into account as progress is made with the solutions of the equations is that of friction. Jeffreys has given much consideration to this matter and he concludes that the tides actually existing on the earth are profoundly affected by frictional forces.

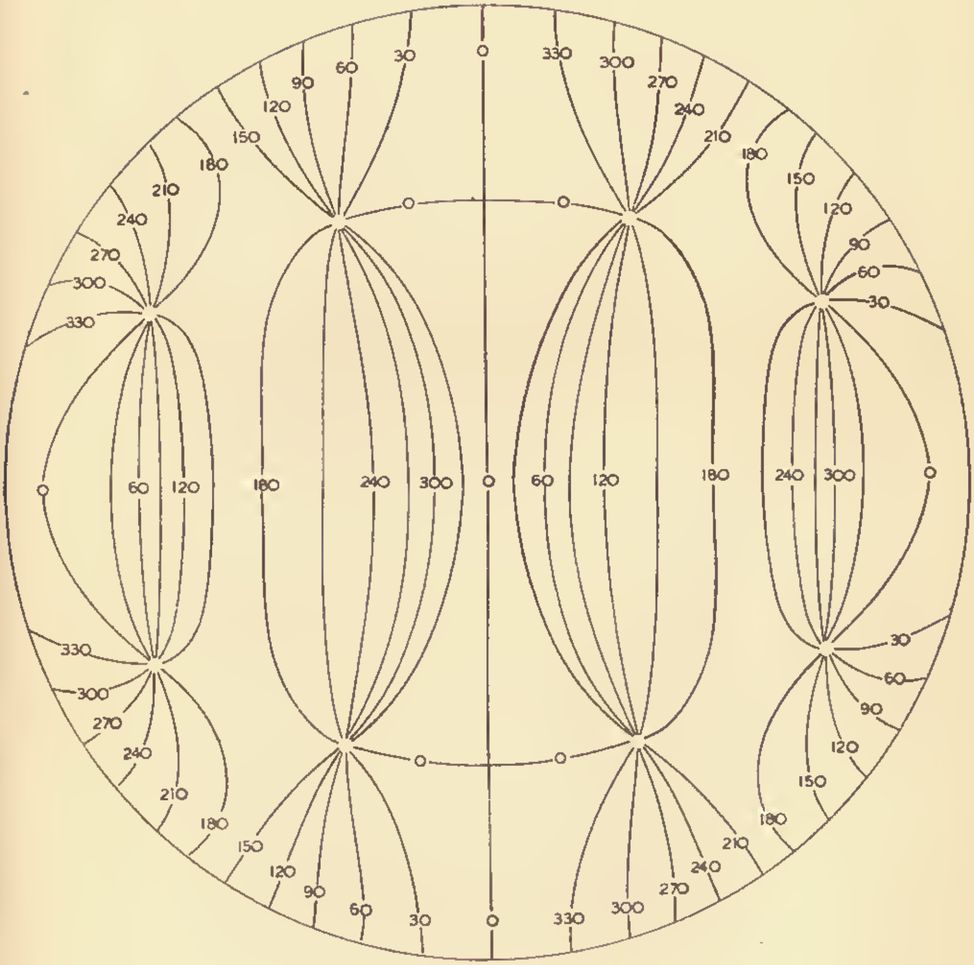

FIG. 23.4. Tides in ocean bounded by meridians 180° apart.
(Depth 14,520 ft.)

CHAPTER XXIV

COTIDAL CHARTS

24.1. Use of deductions from coastal observations

THE theory of tides is peculiar in the sense that the external forces giving rise to them are known with great exactness, and the resultant phenomena can be observed without limit so far as the fringes of the oceans are concerned. Our reasoning powers fail, however, in connection with the intermediate stages and the movement of water in the main oceans. Deductions from the forces have been made up to a certain stage, and it remains to be seen whether deductions backwards from the facts can be carried to such a stage as to make satisfactory alliance with the theoretical deductions.

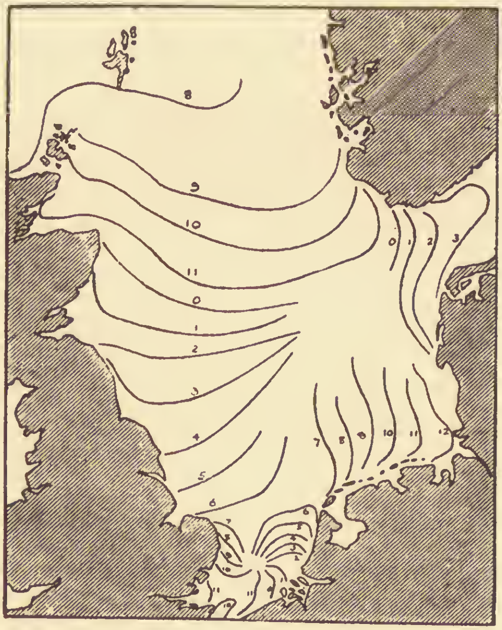

FIG. 24.1. Cotidal lines for North sea, Whewell (1836).

If it were possible to obtain observational data from all parts of the oceans then an accurate view of the propagation of the tides could be given in graphical form. But the difficulties of observing tides of small range in very deep water are almost insuperable, and there is a paucity of information in the central parts of large seas and oceans. Practically speaking, such information only exists where there are oceanic islands.

Systematic collection of coastal data was commenced early last century under the auspices of Dr. Whewell, whose main object was to deduce as far as possible details as to the propagation of the progressive waves then believed to be characteristic of the tidal movements. It is quite reasonable to suppose that if the time of high water is progressively later along a coast then in the *vicinity of the coast* the facts may possibly be interpreted as indicating a progressive wave travelling along the coast. These facts can be interpreted graphically by a system of *cotidal lines,*

which are lines joining all points at which high water occurs at the same instant. These lines, in the instance cited, would be approximately at right angles to the coast.

In the case of a narrow sea, the lines may be drawn across the sea, but naturally in the absence of other information or without some guidance from theory, or by the use of erroneous assumptions, the cotidal lines will vary with the individual who draws them.

24.2. Historical account of cotidal charts

For the tides in waters near Great Britain, the earliest exponent of the art of drawing cotidal lines was Whewell in the year 1833. Later, in 1836, he considerably amplified his first chart, and the results for the North sea are given in Fig. 24.1.

FIG. 24.2. Cotidal lines for English channel and Flemish bight, Airy (1845).

The numbers on the charts are the *cotidal hours*, being the times of high water in hours of lunar time so that the mean tidal period is taken as 12 hours. Thus on a line marked 4, high water is supposed to be four lunar hours after the transit of the moon at Greenwich.

The first attempt made by Whewell was quite modest, as he contented himself by drawing lines more or less at right angles to the coast on the hypothesis of a progressive wave travelling along the coast. In his second attempt he realised that in the Flemish bight the cotidal lines for hours differing by 6 appeared to be continuous, and consequently at a certain point all such lines must meet, and that there must be no tide at the meeting place. On passing through this no-tide point, in any direction, one would pass from high water to low water, or *vice versa*. Such a system was hard of acceptance in those days, and many quite eminent men after him were sceptical of its existence, in spite of the results of observations made by Captain Hewitt, of H.M.S. *Fairy*, in the years 1839, 1840, which definitely showed a very small range of tide in the vicinity of the point indicated by Whewell. Airy in the year 1845 was opposed to the idea of such a no-tide point and proposed the chart shown in Fig. 24.2, which purported to represent two systems of waves travelling

along opposite coasts. What was supposed to take place at the points where the cotidal lines crossed is not explainable.

The amphidromic point indicated by Whewell was to some extent accepted in Tizard's chart, which was included in "Tides and Tidal Streams," published by the Hydrographic Department of the Admiralty in the year 1909, but the amphidromic system proposed is very remarkable; whereas the ordinary amphidromic system is such that the tides appear to revolve round the amphidromic point in the tidal period, in Tizard's chart of cotidal lines the revolution round the amphidromic point is given as two tidal periods; that is, if lines are drawn for each cotidal hour, there are 24 such lines instead of the usual 12 (or at least there would have been 24 if the lines marked 0 had been included). This part of Tizard's chart and also the part for the English channel are illustrated in Fig. 24.3.

FIG. 24.3.—Cotidal lines for English channel and Flemish bight, Tizard (1909).

It will be noted that Tizard's chart, though it did not exactly specify a no-tide point, implies the existence of such a point. Yet even Darwin in his article on tides for the Encyclopædia Britannica in 1911 included an illustration of a tidal chart which rejected such an amphidromic system and utilised Airy's conception of two waves. The chart he gave was taken from Berghaus, Physicalische Atlas.

Another feature of the early charts is exhibited in the illustration of Airy's chart for the English channel (Fig. 24.2). The cotidal lines are very much curved inwards to the channel, so much so that for some of the lines the effect is to suggest that it is high water in the centre of the English channel when it is only half tide at the sides. This curvature is based upon the supposition that the tide will be earlier in deep water than in shallow water, seeing that the rate of travel of a progressive wave increases with the square root of the depth. It will be noted that it is the idea that tides can be represented by progressive waves that is really responsible for the error, for we have shown that where there are boundaries then the tide must be more nearly represented by standing oscillations.

We can consider the fallacy of the supposition as in Art. 21.5, but it is sufficient to point out that if we consider the gradient from the sides to the centre of the channel, then if there is much curvature in the cotidal lines there must be streams

transverse to the channel, and the rates of these streams must be comparable with those along the channel. Yet in such charts as those given in Fig. 24.2, the effect of the curvature, as already pointed out, is to suggest that it is high water in the centre of the English channel when it is half tide at the sides! If such a state could occur, the tidal streams transverse to the channel would be about equal to those along the channel, and there is no evidence of such strong tidal streams flowing north and south in the English channel, with phase relations to produce the supposed gradients.

A mathematical examination of this problem was made by Proudman, and he found that for a progressive wave in a narrow channel the curvature of cotidal lines is very small indeed. He dealt with variable depth, and concluded that when the breadth of the channel is only a fraction of the wave-length then the curvature of the cotidal lines is always very slight.

It is perhaps unnecessary to enter into further details of the many charts which have been proposed within the last century, not only for the seas around the British

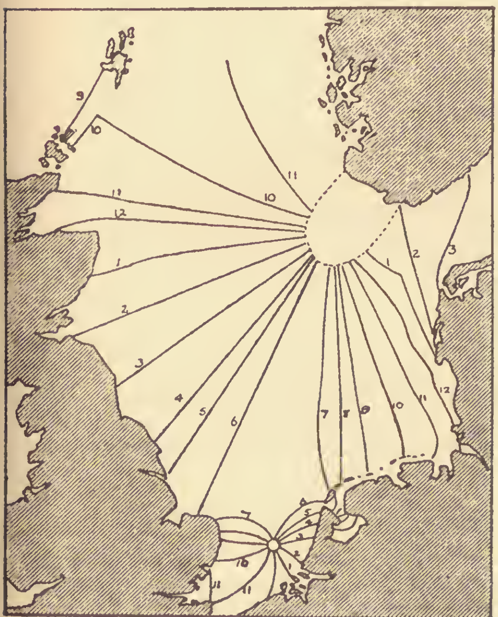

Fig. 24.4. Cotidal lines for North sea, Harris (1904).

Fig. 24.5. Cotidal lines for North sea, Sterneck (1920).

Isles, but for the whole world. Harris, of the United States Coast and Geodetic Survey, made a most elaborate attack on the problem. The chart he proposed for the North sea is illustrated in Fig. 24.4. His conception of oscillating areas was applied to the whole of the oceans, but his charts cannot be accepted as accurate. His theory neglects gyroscopic effects and so is fundamentally incomplete. The work of Defant and of Sterneck in recent years deserves mention as they have exploited mathematical methods with marked success in the cases of many narrow seas where the tidal streams transverse to the length of the sea are comparatively feeble. Sterneck has also given charts for the whole world, but the principles on which they are drawn are rather obscure. His chart for the North sea (Fig. 24.5) agrees in a general way with a recent chart prepared for the Admiralty, which will now be considered.

24.3. Deductions from tidal streams

It was indicated in the previous article that some of the cotidal lines suggested for the English channel were not likely to be correct because the tidal streams required to maintain the proposed distribution of elevation were not in agreement

with those actually existing. It does not appear as though criteria such as these were adequately considered by authors of cotidal charts until a few years ago. Such efforts as they made ignored all but coastal data of elevations, and the very valuable and extensive observations of tidal streams were ignored. Yet these observations of tidal streams have been taken at places where direct observations of the elevation of tide are lacking on account of the great observational difficulties.

In Chapters XVII and XVIII the relations between elevation and tidal stream were discussed at some length in connection with progressive and standing oscillations, and it has been generally indicated that mathematical formulæ can be given, relating tidal streams to elevations, including also the gyroscopic effects. If the tidal stream is known at any place, then these formulæ give the surface gradient; that is, the rate of change of the elevation with the distance. In practice, of course, we should compute the gradient in two standard directions at right angles. These calculations are very simple when the formulæ have been established, and immediately the criterion can be applied to any chart of cotidal lines: Do the indicated surface gradients conform to the gradients computed for the tidal streams?

As a small example of this procedure, consider the progress of the tide down the east coast of England. From the data given in Admiralty publications we find that off Hartlepool, for instance, the stream is at its maximum approximately at high water; that is, the relation between tidal elevation and tidal stream is that pertaining to a progressive wave travelling to the south along the coast. Consequently we would deduce that the cotidal lines should be approximately at right angles to the coast. More elaborate calculations confirm this view, so that Tizard's conception of cotidal lines at an acute angle to the coast was definitely erroneous. Where cotidal lines are nearly parallel to a coast, the principal part of the tidal stream must be towards the coast and away from it, and there is no evidence of such régime of tidal streams off this coast.

Similar considerations all round the coasts provide valuable data enabling the directions of the cotidal lines near the coasts to be stated accurately.

Out at sea, where we have no observational data to give high water times, these considerations cannot yield information as to the actual direction of the cotidal lines but they can be used to check whether the surface gradients indicated by the suggested chart are in strict conformity with the tidal streams.

24.4. The construction of modern cotidal charts

The relations between tidal streams and surface gradients have so far been utilised in the way of criticism of any proposed chart, but we shall now go further and show how these relations can be used in a more positive way in order to produce cotidal charts. We shall not attempt to prove the mathematical formula, but the following equation gives some indication of the formula used.

Let a line be taken from coast to coast, and let the tidal stream in this direction be denoted by u. The tidal stream in general will have two components at right angles; let the second component be denoted by v in a direction to the left of the direction in which u is measured. Then the mathematical formula can be expressed as follows:—

$$\begin{aligned}(\text{Rate of increase of } u \text{ with time}) &- 2\omega v \sin l \\ &= -g \times (\text{surface-gradient}).\end{aligned}$$

Here g is the coefficient of gravitational force, ω is the angular rate of revolution of the earth, and l is the latitude (as in Arts. 20.2 and 20.4).

We are in a position to apply some simple tests to the formula before accepting it. It can be noted that in a narrow canal, with our principal direction taken along the canal, the value of v must be negligibly small. We then see that when the velocity along the canal is at its maximum, then its rate of increase is momentarily zero, and therefore the surface gradient must be zero. This can happen either if the elevation is also at a maximum (which is the case corresponding to a progressive wave) or if the whole surface is momentarily flat (which is the case of a standing oscillation). Hence this simple test verifies the relation between u and the surface gradient. Now

let us examine the meaning of the other term. It asserts that the effect of the earth's rotation upon the tidal stream v can only be balanced by a surface gradient increasing to the right of v (that is, in the direction of u). The gyroscopic forces are greatest at the pole and zero at the equator, hence the introduction of the sine of the latitude.

Such is the fundamental formula connecting tidal streams with surface gradients in small seas. It embodies principles which we have adequately discussed, and it is quoted simply to indicate the basic principles. The formula as it stands is only applicable for small seas; the astronomical forces are left out of account because they change relatively very slowly within the area of a small sea.

Now suppose that the tidal elevations are known at a given time at a place X on one coast and a place Y on another coast, and that at certain places A, B, C, D . . . on a line between the terminal points X, Y, we also know the tidal streams at the

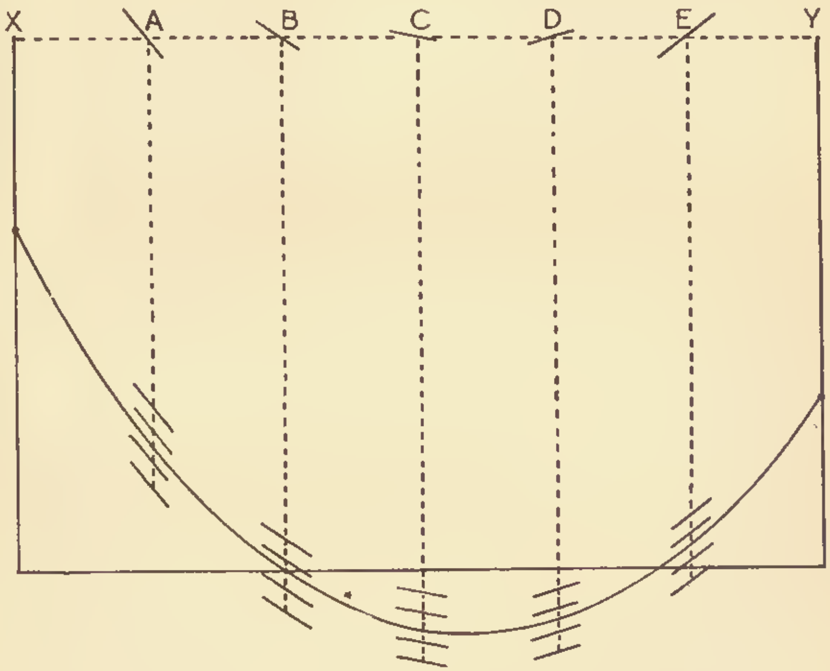

FIG. 24.6. Calculation of surface gradients from coast to coast.

same time. Our knowledge will necessarily be derived from the reductions of observations made at different times, and the mean lunar constituent (M_2) derived from such reductions will be taken. We shall suppose that the data are manipulated so as to give elevations and streams at the time of transit of the mean moon at Greenwich. At this precise time, therefore, we shall know the mean lunar tidal stream at stations A, B, C, D . . . and the elevations at the terminal points X, Y. From the tidal streams we can compute the surface gradients. Now in Fig. 24.6, at the sides X, Y, draw lines vertically upwards, representing the elevations of tide at these two places, and corresponding to the point A draw a number of parallel lines having a slope giving the surface gradient as computed for A. Similarly draw a number of parallel lines for B with the slope pertaining to the station B, and so on. Our problem is to draw a curve having the right elevations at X, Y, and the right gradients for A, B, C, D. Such a curve is drawn in the figure, and only one curve (within small limits) can be so drawn. Hence this curve gives the elevations (at the chosen time) for all points across the section XY.

If we repeat the work for one other time, say three hours later, we can get for any required place on the section the range of tide as well as the time of high water,

and if this work can be effected for a number of sectional lines across a channel, we get a complete picture of the tides.

Again, we can draw lines more or less transverse to these sectional lines and we can verify that along these lines also the surface-gradients are in conformity with the observed tidal streams. Such a network implies the utmost stringency and we can feel quite confident that the results are as accurate as possible—they conform to the coastal data as well as the data of tidal streams in every direction, and so satisfy all the requirements. It may be necessary to allow a little for errors of observation, but the fitting of the network of stations is equivalent to the smoothing of observational data, with enhanced accuracy in the results.

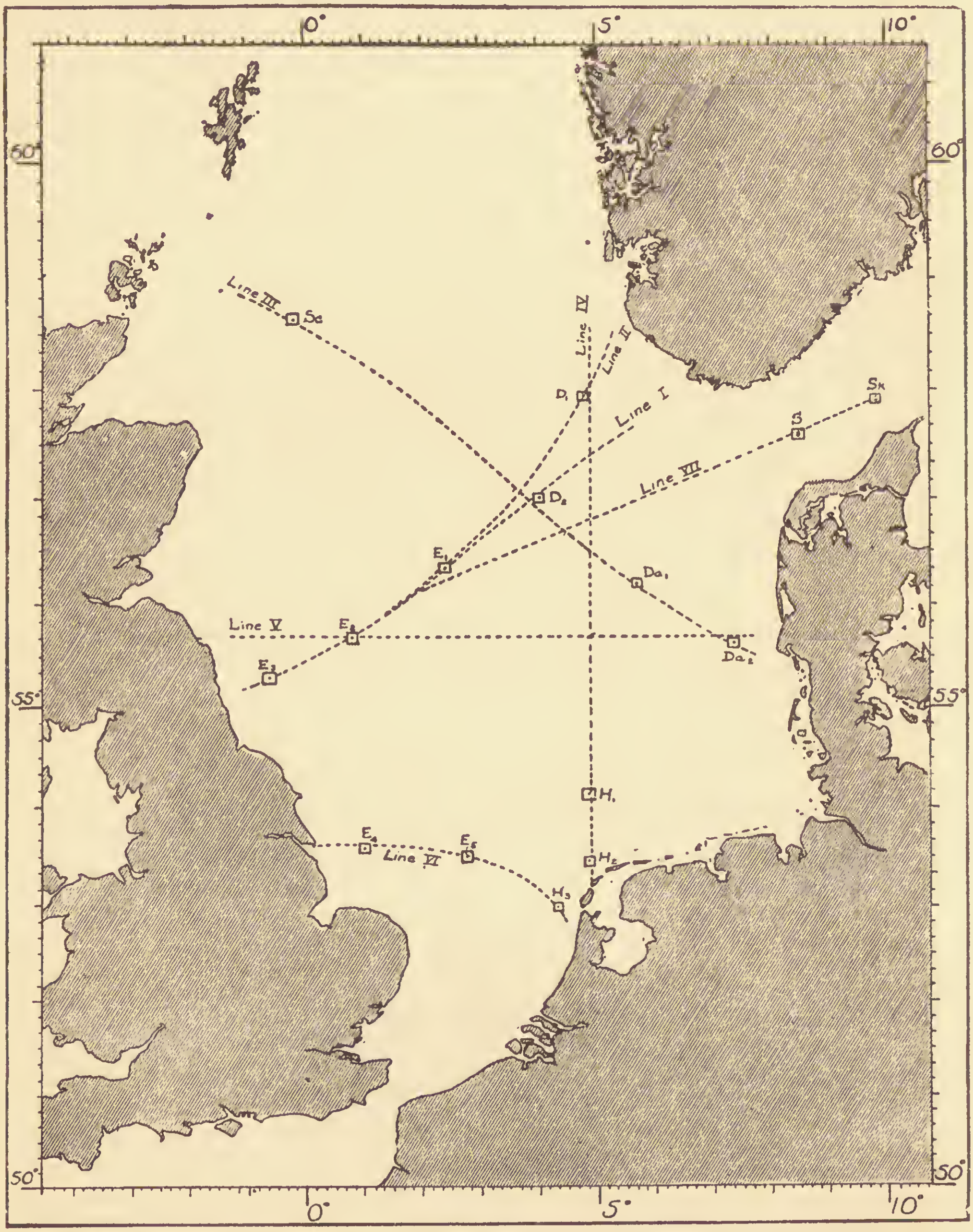

Fig. 24.7. Network of stations in North sea used by Proudman and Doodson (1922–1924).

24.5. The Admiralty chart of cotidal lines

The principles discussed in the previous article were first applied by Proudman and Doodson for the North sea in the year 1922, using a network of stations (Fig. 24.7) at which observations of tidal streams had been taken, under the auspices of

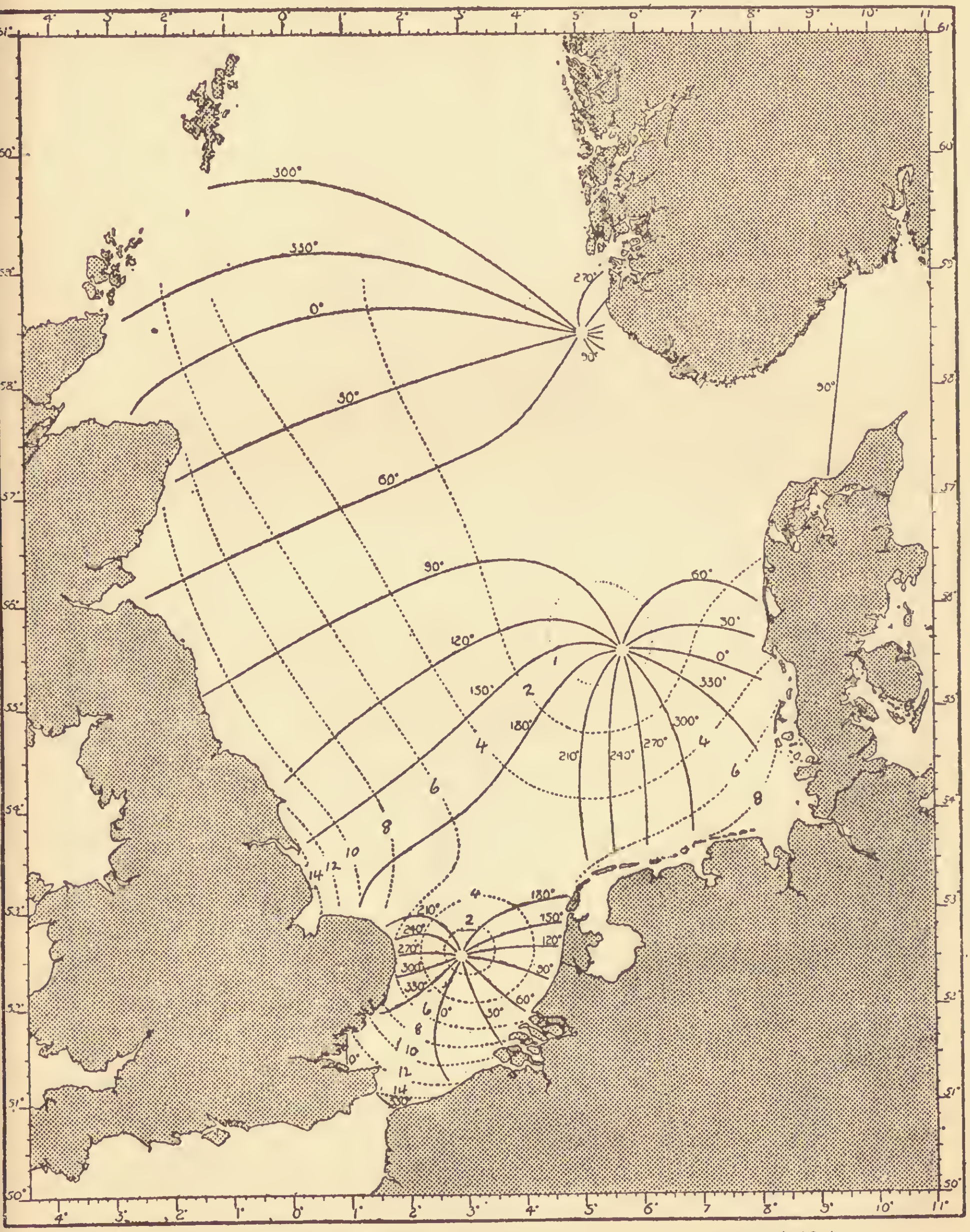

FIG. 24.8. Cotidal lines for North sea, Proudman and Doodson (1924).

the International Council for the Exploration of the Sea, about ten years previously. The results were so promising that the Hydrographic Department requested the work to be extended and they provided a large amount of supplementary data, some being actual observations of tidal elevations as taken by submarines out at sea. The observations, though diverse in origin and character, all fitted together in a very

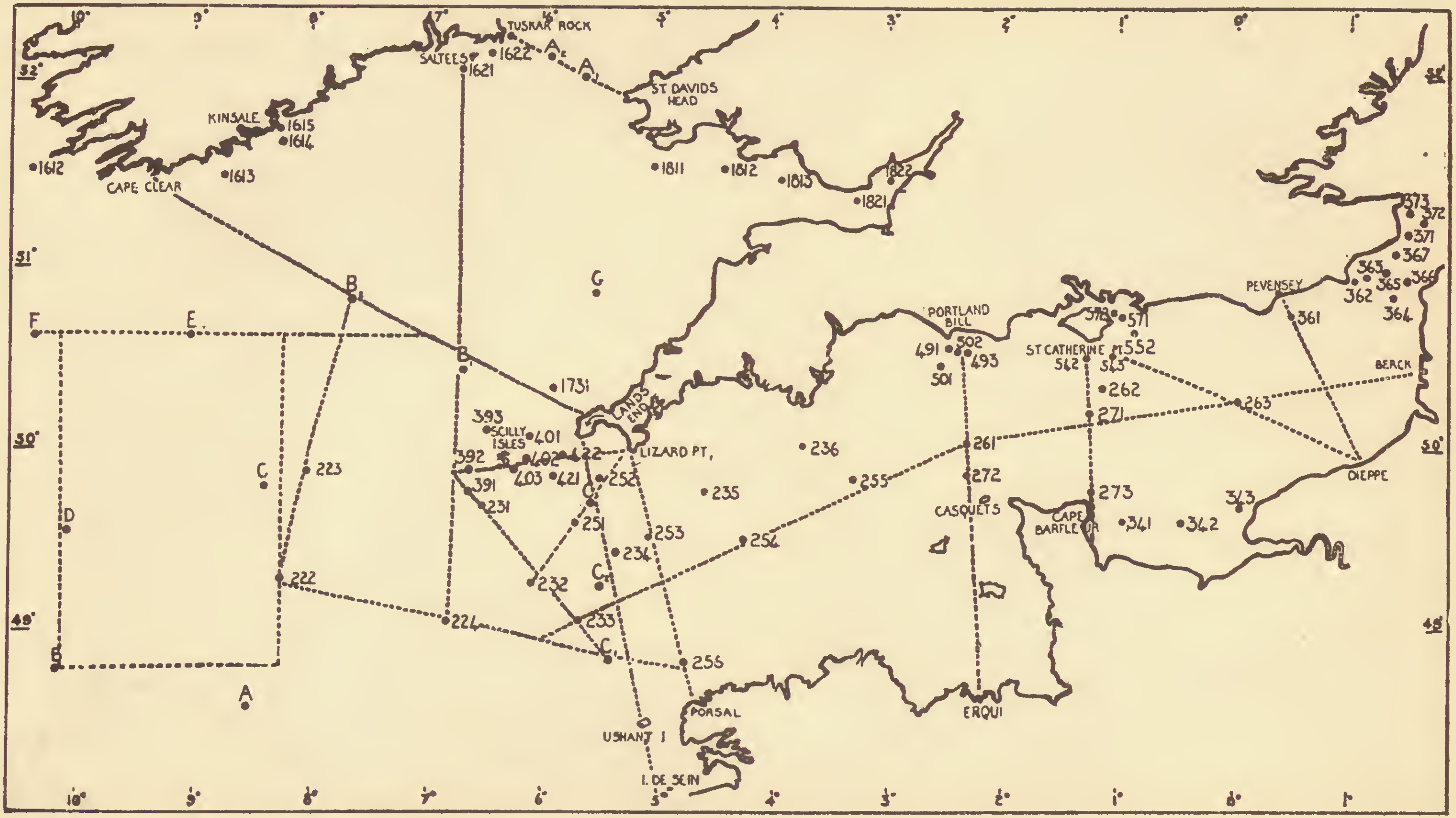

FIG. 24.9. Network of stations in English channel, Doodson and Corkan (1931).

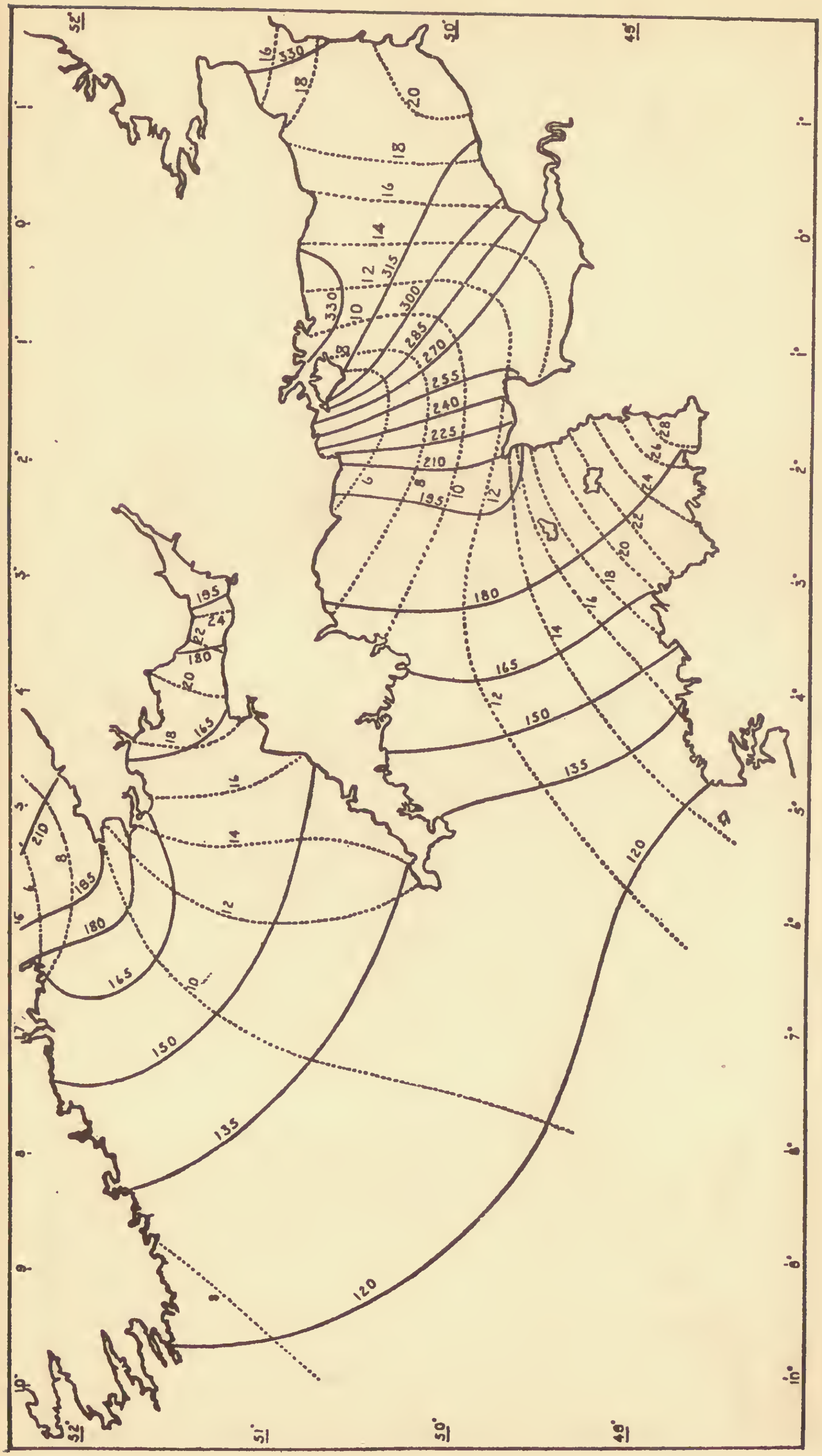

FIG. 24.10. Cotidal lines for English channel, Doodson and Corkan (1931).

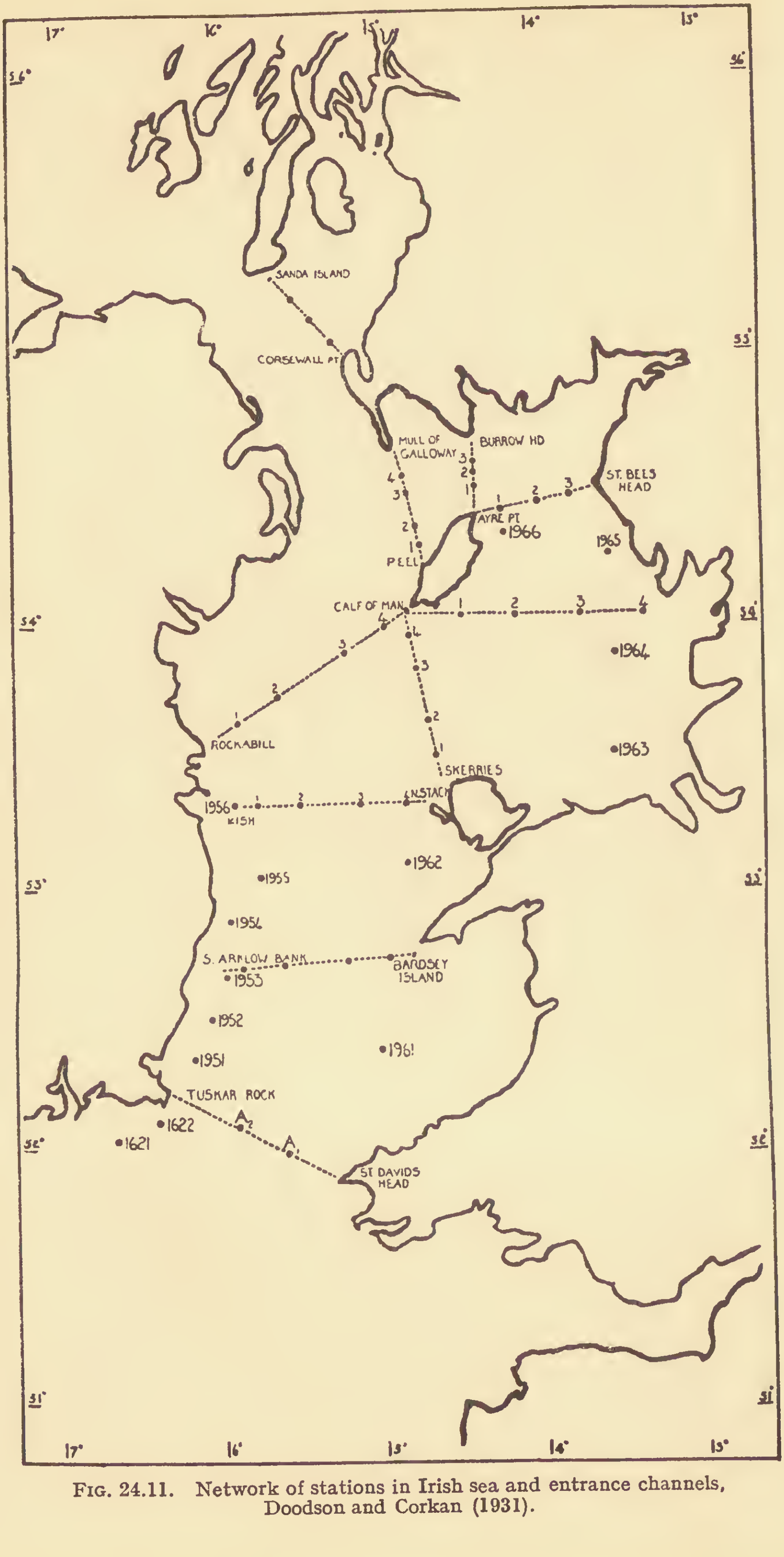

FIG. 24.11. Network of stations in Irish sea and entrance channels, Doodson and Corkan (1931).

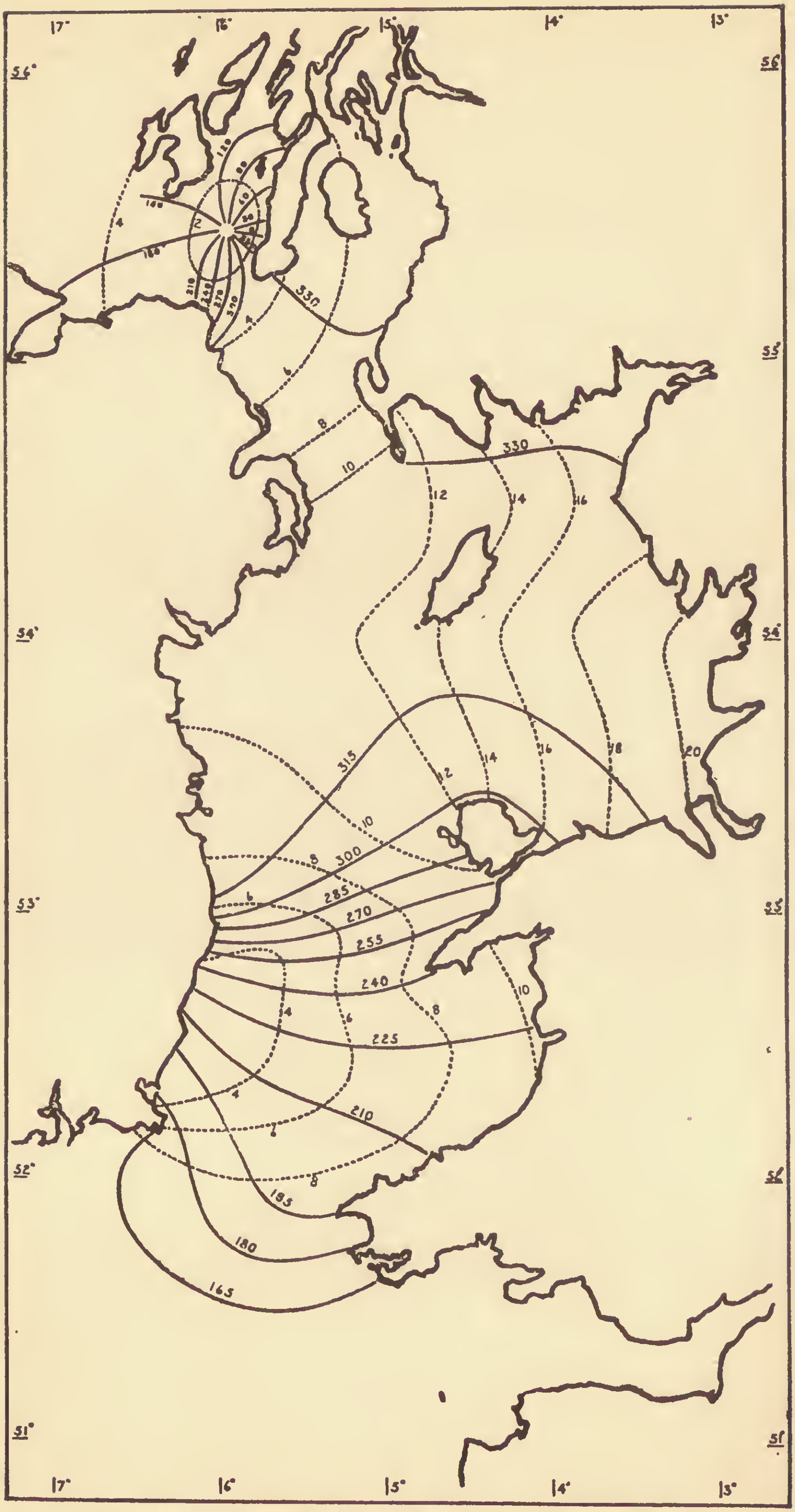

FIG. 24.12. Cotidal lines for Irish sea and entrance channels, Doodson and Corkan (1931).

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convincing manner, and a chart was prepared, as shown in Fig. 24.8, in which the cotidal lines are shown at intervals of 30° in the phase-lag (g) of the average tide (M_2), corresponding to intervals of lunar hours after the transit of the moon at Greenwich, and the ranges at intervals of 2 ft. are shown by the broken lines. On the basis of this chart the Hydrographic Department prepared a chart for official use.

In tbe year 1931, the Department requested the Liverpool Observatory and Tidal Institute to extend the charts to cover the English channel and Irish sea, and the work was carried out by Doodson and Corkan. The elaborate network of stations utilised by them is illustrated in Figs. 24.9 and 24.11, and the resulting charts are shown in Figs. 24.10 and 24.12. The cotidal lines are drawn at intervals of 30° in tbe phase-lag (g) of the average tide (M_2), that is, at intervals of hours of lunar time after the transit of the mean moon at Greenwich. The broken lines are the co-range lines, with the ranges of tide in feet. It may be remarked that some of the observations used for constructing these charts were made a century ago, but they fitted in with the more recent observations, some of which, at the entrances to the channels, were taken specially for this work.

On the basis of these charts a new chart, now No. 5058, was prepared and published by the Hydrographic Department, in which the time intervals are given in solar hours and minutes, and other figures enhancing the value of the charts in navigation are also included. The complete chart is shown on a much reduced scale in Fig. 24.13; for further explanation the chart itself should be consulted.

These charts, it must be emphasised, are not dependent on any assumed hypothesis as to the theory of the propagation of the tide. They are based upon actual observations over the sea as well as on the coasts, and the well-established mathematical relations between stream and surface-gradient. The success of the method in fitting together diverse observational data and the general consistency of the network is a testimony to the soundness of the methods.

It will be noticed that in the English channel the cotidal lines are not convex inwards like the earlier charts; in fact, if anything, they bend slightly outwards towards the ocean. The same holds in the channels between Great Britain and Ireland. In the North sea we have the phenomena of two well-defined amphidromic systems, one in the Flemish bight and one west of Denmark and another system, not so well defined, off Norway. The tides near Norway are so very small that a little doubt necessarily exists as to the reality of this point. If it exists at all it is very near the coast; it may be that it does not actually exist and that the cotidal lines converge to a point inland. Such a system is said to be a degenerate amphidromic system, and examples are to be found near south-east Ireland and near the Isle of Wight.

FIG. 24. 13. Reproduction of Admiralty Chart No. 5058.

24.6. Charts for the oceans

It has been shown that the difficulties of drawing cotidal charts for narrow channels and small seas are very great, and it is therefore hardly to be expected that at the present time much confidence can be expressed in any sets of charts for large oceans. Airy, Harris, and Sterneck, among others, have attempted the solution of the problem for the whole world. The principles utilised have been very diverse. Perhaps Sterneck's charts are the most credible, but where the basis of a chart is

FIG. 24.14. Cotidal lines, Atlantic ocean, Harris (1904).

speculation in a high degree, and where the complications are very great, as in the Indian and Pacific oceans, it is not considered wise to reproduce such charts. The Atlantic ocean, however, is a comparatively simple basin, and we therefore reproduce in Figs. 24.14 and 24.15 the charts made by Harris and Sterneck. There is a large measure of agreement in the general picture of the motion.

The distribution of cotidal lines for the diurnal tides in the oceans must be very different from that of the semidiurnal tides, and separate charts for semidiurnal and diurnal tides are necessary. This arises from the large differences in the speeds of the tidal motions, and consequently, since speed is a factor, we must expect slight differences between the distribution of lines for two semidiurnal constituents of

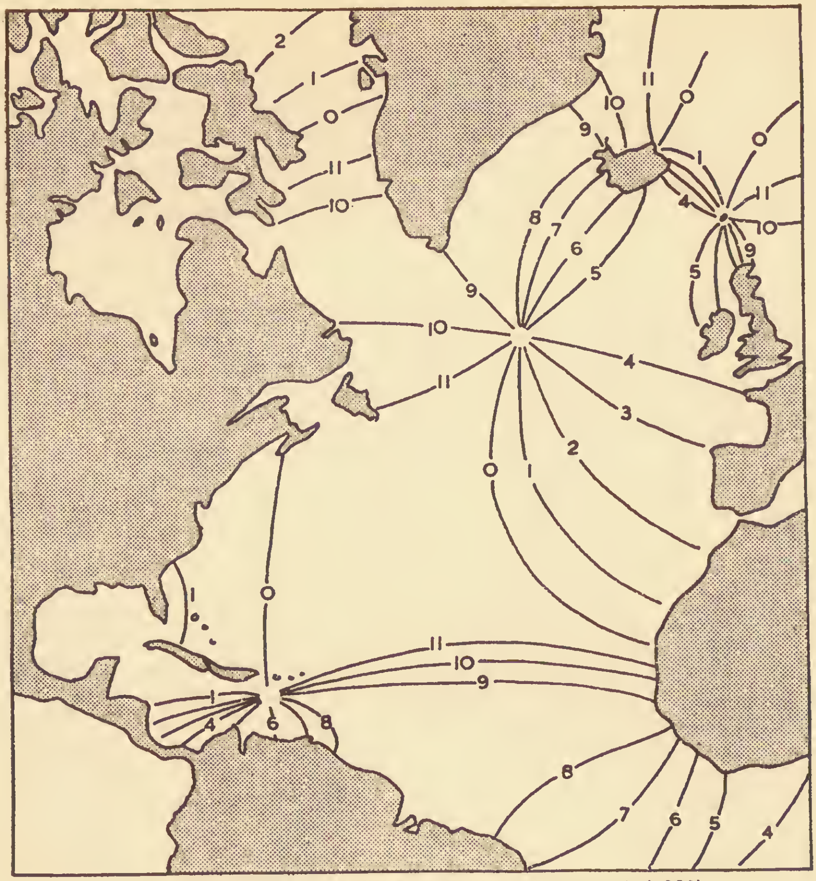

FIG. 24.15. Cotidal lines, Atlantic ocean, Sterneck (1920).

nearly equal speeds, such as S_2 and M_2. Hitherto it has been regarded as sufficient to draw charts either for the mean lunar constituent M_2 or for the compound constituent $M_2 + S_2$, but the latter practice is not ideal.

CHAPTER XXV

EXPLANATIONS OF TIDES ROUND GREAT BRITAIN

25.1. Explanations of tides in British waters

In this chapter we shall give explanations of the movements of the tides in the seas around Great Britain. The explanations are limited to the areas for which accurate charts of cotidal lines are available, so that there is nothing speculative about the explanation. The principles have already been explained in detail. The explanations are adapted from those given by Proudman, in 1923 for the Irish sea and in 1924 for the North sea. They are given only for the mean lunar semidiurnal tide (M_2).

25.2. The tides in the Irish sea

The tides in the Irish sea are comparatively simple. The basin is nearly land-locked, and the water enters by the North channel between Ireland and Scotland, and by the South channel (St. George's channel) between Ireland and Wales. If the basin were completely enclosed, the tides would be inappreciable, so that the large tides experienced in the Irish sea are due to the influx and reflux of water from the Atlantic ocean. If, therefore, we know only the tidal streams through the two entrances it should be possible to deduce the tides in the sea.

The tidal stream entering by the North channel is a maximum inward about eight lunar hours after the transit of the moon at Greenwich ; the actual interval is more nearly 7 hours 50 minutes, but it is more convenient to refer to the round figure. We use lunar time because we are discussing the mean lunar semidiurnal tide, with a period of 12 lunar hours. The tidal stream entering by the southern channel reaches its maximum at about the same time (hour 8). These streams will turn back and begin to run outwards three lunar hours later or at hour 11. At this moment there will be the maximum amount of water in the basin.

It will be realised that we are bound essentially to have a standing oscillation, with zero streams at high water. The tidal streams can be considered as building up the elevations and so producing surface gradients which ultimately reduce the streams to rest. The high water elevations will thus increase from the two entrances to the coasts furthest removed from them.

It will be noticed that according to this simple explanation high water would occur simultaneously throughout the Irish sea at hour 11. From the chart of cotidal lines for the area given in the previous chapter we see that this is approximately so, for high water occurs throughout the sea within half an hour of the average time. We also see that the range of the tide increases according to the explanation.

In a straight channel, the co-range lines, the lines of equal range, will run transversely to the channel, or, what comes to the same thing, transversely to the lines of the flow of the stream. Considering the Irish sea as an irregular channel, the co-range lines should run transversely to the lines of flow of the tidal streams. It would require very little imagination on our part to construct the stream-lines, but the flood-lines were obtained by Beechey in 1848, and we may therefore take the co-range lines to be transverse to these.

Now consider the effects of the gyration of the earth, which have so far been neglected. The tidal streams are strongest everywhere at hour 8, in accordance with the theory of standing oscillations (Chapter XVIII), and the effect of the gyroscopic forces is to cause an acceleration to the right of the streams. Then we must either have a turning to the right of the flood or else a surface gradient rising to the right of the flood. This gradient would therefore have its maximum at hour 8.

It is possible to calculate what this gradient should be, and actually it is about three-quarters of the primary gradient along the stream lines. Hence we have a tendency for the water to be elevated on the coast of north Wales relatively to the surface on the coast of south Scotland, at hour 8. The elevation and depression due to this gyroscopic gradient thereafter diminish while the elevation due to the primary gradient is increasing. Thus on the Welsh coast there are two oscillatory movements to combine, and near the time of the conjoint high water one of these movements is causing the elevation to fall while the other is causing it to rise, so that the joint effect is to make high water somewhat earlier than it would have been if there had been no gyroscopic forces. Hence high water is earlier than hour 11 on the Welsh coast and from similar reasoning we see that it is later than hour 11 on the Scottish coast. This is verified by the cotidal chart.

The effects of friction can also be discussed in this elementary manner, and a further approach to a complete explanation can be thus achieved.

It will be noticed that on the coast of south-east Ireland the tidal range is small and the range increases across the channel. In the absence of gyration of the earth there would be a nodal line across the channel from this point, in accordance with the theory of standing oscillations. The effect of the gyroscopic forces, however, is to cause an elevation on the right of the channel at the time when the surface would otherwise be level, as was shown in Art. 22.2. This would lead us to expect an amphidromic point in the centre of the channel. The cotidal lines, however, appear to radiate from a point inland, and the reason for this degeneration appears to be that we have not got a true standing oscillation even in the absence of gyration. It is shown in Arts. 18.3 and 18.4 that a standing oscillation in a channel closed at one end may be considered as the combination of an ingoing wave and of a reflected wave. If, however, there is friction in the sea, this would reduce the elevation of the ingoing wave and still more the outgoing wave, so that the two are not equal in amplitude as in the theory of simple standing oscillations.

In the North channel an amphidromic system exists and is situated as shown in Fig. 24.13; the explanation of this system depends upon similar principles. The position of an amphidromic point depends upon the depth and the configuration of the basin, but if the configuration is not of a simple or geometrical type then simple detailed explanations cannot be expected.

Many smaller features of the tidal motion could be discussed and explanations tendered, but it should be sufficient to have shown that the main features are adequately explained by the principles utilised above.

25.3. The tides in the North sea

Let us firstly suppose that a barrier is placed across Dover strait, and that the earth's gyration is ignored. The tides in the North sea will then be maintained wholly by the oceanic oscillations to the north of Scotland, for the sea is too small for the attractive forces of the moon and sun to have any large direct effects within the sea. The result is that we have an approximately rectangular channel for the North sea, with a smaller channel for the Flemish bight, at the closed end. The theory of oscillations in closed channels (see Art. 20.4) has already been discussed, and indicates that the tidal oscillations will be of the type called standing oscillations. The depth of the sea is such that the nodal lines (that is, the lines of zero range) will occur approximately as indicated by the full lines in Fig. 25.1. There are three nodal lines, and between any two lines there will be high water (or low water) simultaneously, with a maximum range half-way between the lines and zero range on the lines. Thus on crossing a nodal line we pass from a region of high water to a region of low water, or *vice versa*.

These lines are fixed by the depth of the sea and give four areas. From the Atlantic oscillation north of Scotland we see that in area 1, the most northerly of the four areas, low water will occur approximately three hours after the time of lunar transit at Greenwich (that is, at hour 3 in lunar time), and there will be high water in area 2, low water in area 3, and high water in area 4. These are denoted by the letters H and L.

At this time (hour 3), the tidal streams will be everywhere zero, as we saw from the theory of standing oscillations, and three hours later when the surface is level

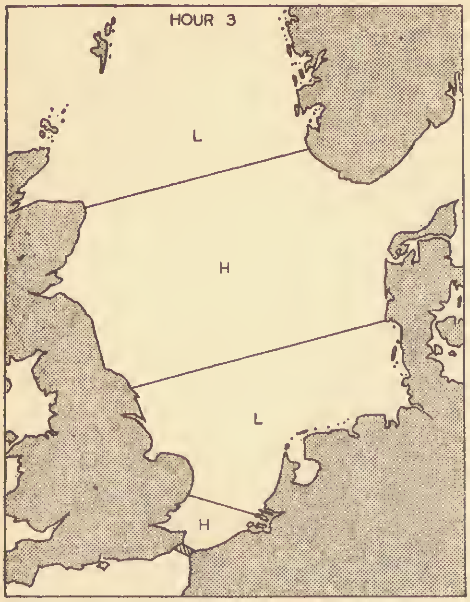

Fig. 25.1. North sea : first phase of standing oscillation in absence of gyration.

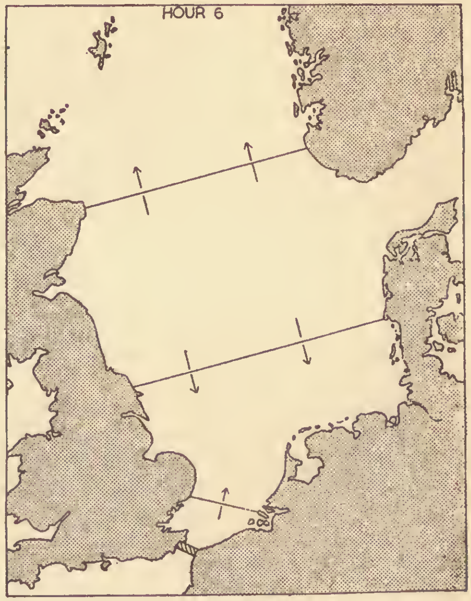

Fig. 25.2. North sea : second phase of standing oscillation in absence of gyration.

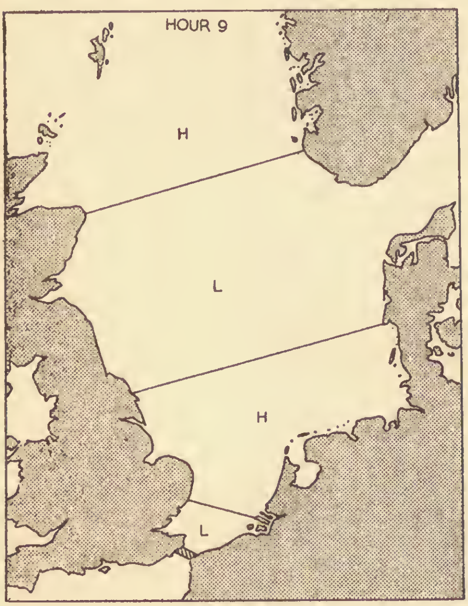

Fig. 25.3. North sea : third phase of standing oscillation in absence of gyration.

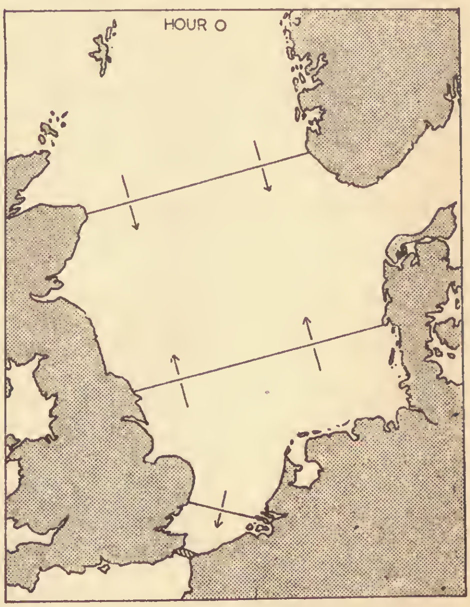

Fig. 25.4. North sea : fourth phase of standing oscillation in absence of gyration.

they will be at their maximum strengths, and the greatest streams will be across the nodal lines. Thus the successive stages at hours 3, 6, 9 and 12 can be pictured as in Figs. 25.1 to 25.4. At hour 12, of course, the conditions are the same as at hour 0.

Now consider the effects of the earth's gyration (again see Art. 20.4). As we have seen, the gyroscopic forces build up a gradient transverse to the tidal currents, and thus at hours 6 and 12 there will be gradients along the lines which were nodal

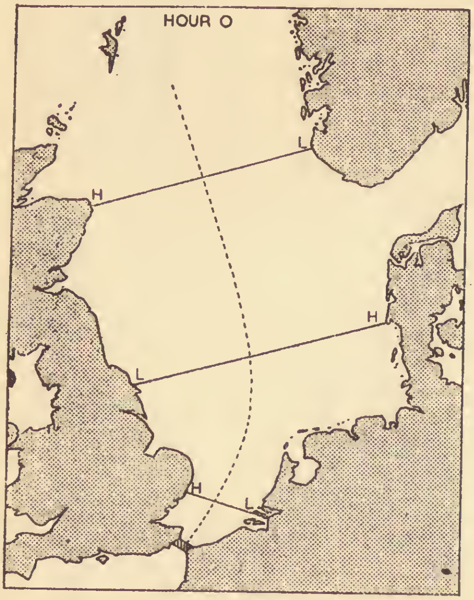

FIG. 25.5. North sea : first phase of standing oscillation, modified by gyration.

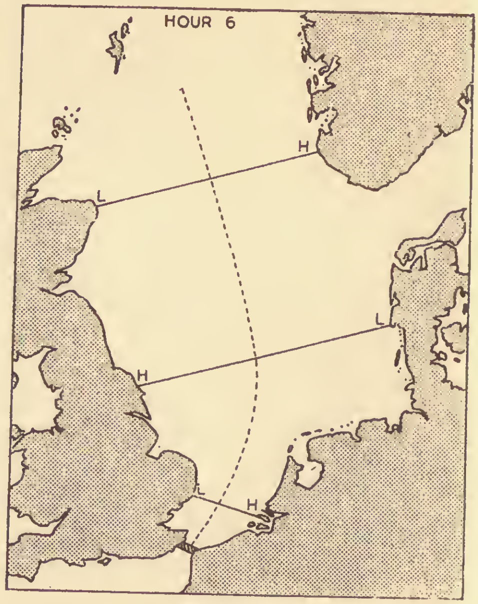

FIG. 25.6. North sea : second phase of standing oscillation, modified by gyration.

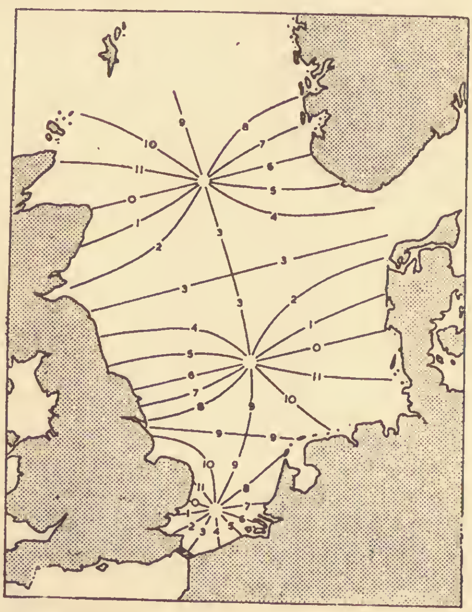

FIG. 25.7. North sea : cotidal lines deduced from particular phases of standing oscillation, modified by gyration.

lines in the absence of gyration. On the line between areas 1 and 2, at hour 12 (or 0), there will be a gradient to the west, so that the level is higher on the western coast than it is on the eastern coast. Similar considerations lead to the determination of the gradient along each of the nodal lines at hours 6 and 12, and the resulting

high and low waters are denoted by H and L in Figs. 25.5 and 25.6. Consequently, there are brought into existence tidal oscillations on the original nodal lines, but there will be points half-way along the lines at which the level will still remain unchanged. In other words, the original lines of zero tidal range have been reduced to three points, one on each line, at which there is no tide.

If we trace out the sequences of high water we can sketch the cotidal lines given in Fig. 25.7 and we see that there are three amphidromic systems.

We have yet to consider the effects of friction, for in this relatively shallow sea tidal friction is of importance. The tidal oscillation pictured in Fig. 25.7 progresses round the shores of the sea in an anti-clockwise direction (see Art. 22.2). Tidal friction causes a loss of energy which leads to a diminution of the range of tide as the wave progresses. Consequently, instead of there being equal and opposite conditions on the eastern and western shores, the tide on the eastern shore is much less than on the western shore, and the point of zero range is moved towards the east. In the case of the most northerly point, the shift is so great that there is an element of doubt as to whether the amphidromic point actually exists—if it does exist it is very near the coast, or there may be a degenerate amphidromic system with the lines converging to a point inland.

In the Flemish bight, this theory, with a barrier across Dover strait, gives the higb water at the strait at about hour 3. If we remove the barrier the result is to swing the amphidromic system round, but the influx of energy through the strait is relatively small so that the changes in the distribution of cotidal lines are inappreciable outside the bight.

25.4. The tides in the English Channel

It is not necessary to enter into much detail concerning the tides in the English channel. It is clear that if a barrier were placed across Dover strait, then in the absence of the earth's gyration the Atlantic tides would maintain a standing oscillation, and the depth is such that the nodal line, transverse to the channel, would pass near the Isle of Wight.

The gyroscopic forces do not in this case cause an amphidromic point to develop on the nodal line, but that there is a tendency for such a system to be developed is apparent from the converging of the cotidal lines towards the Isle of Wight, so that we get a degenerate amphidromic system with the lines converging to a point inland, north-west of the Isle of Wight.

CHAPTER XXVI

DOUBLE HIGH AND LOW WATERS

26.1. The phenomena described

In certain parts of the world, notably in the English channel, there is a special class of phenomena in which there are double high or low waters. The phenomena are of the type illustrated in Fig. 26.1 by the full line for the case of double high waters. The low waters are then approximately of the normal character but the high waters show two peaks with a shallow trough between them. The peaks are generally of unequal height and there may be a slow change in the inequality during a lunation, so that sometimes the first peak is greater than the second, and sometimes it is less.

The phenomena of double low waters is of exactly similar character, but to avoid confusion we shall discuss only the double high waters.

26.2. Erroneous explanations

A very notable case of double high waters is that occurring at Southampton and many erroneous explanations exist as to the cause of the phenomena. Thus it is commonly asserted that the phenomena are due to the situation of the port behind the Isle of Wight, and the first high water is said to be caused by the tide reaching Southampton from the west *via* the Solent, while the second high water is said to be due to the tide which has travelled round the Isle of Wight and reaches Southampton from the eastward. The time difference of two hours is supposed to be accounted for by the extra time taken by the second wave in its longer journey. The exponents of this theory do not explain how it is that the low waters are not similarly affected, and it is clear that the phenomenon of arrival of the peaks does not differ from that of the arrival of the troughs. Neither does the theory account for double low waters at Portland, which is not sheltered by the Isle of Wight.

The above explanation has been derided, and its fallacies exposed, by eminent men for over a century, including such men as Lord Kelvin and Sir George Darwin, but the "explanation" persists and is given regular currency.

Another "explanation" which does not depend on the presence of the Isle of Wight, and so is said to be applicable to Portland as well as to Southampton, is that the first high water at Southampton is the direct one from the Atlantic up the Channel, while the second one is due to the wave which is supposed to have travelled round Scotland and enters the Channel through Dover strait. It is clear, however, that this theory is based upon the same fallacious reasoning as the first theory, the underlying assumption being that two waves act independently of one another and produce separate high waters.

General reasoning alone should be sufficient to expose the fallacy. The tidal motion is a wave of long period, but the laws of combination of waves are the same whatever be the periods, so that we can consider the analogies from light waves and sound waves. Two pure waves produce a combined wave of the same period. Thus two light waves of the same period have the same colour, and if they occur together and are mingled the colour remains unchanged. Two violins producing a pure note of the same period will not result in the listener hearing a note of half the period. So also two tide waves of the same period will continue to produce a tide of the same period ; in a given interval of time, therefore, the resultant tide will have precisely the same number of high waters and low waters as each of the constituent waves.

It is a very simple matter to draw two curves representing tides caused by waves arriving at different times, and to combine the two by adding the ordinates. The

same proof follows quite simply by trigonometry. Let the two waves arrive at times $-$ T and $+$ T measured from the mean time of arrival; then they can be represented by

$$A \cos n(t + T) \quad \text{and} \quad A \cos n(t - T)$$

if both have the same amplitude A, and the same speed n. The sum of these waves is

$$(2A \cos nT) \cos nt$$

so that the amplitude of the resultant wave is 2A cos nT, but it has exactly the same speed n as the constituent waves. Hence the combination of two normal tidal oscillations will never produce double high waters or double low waters.

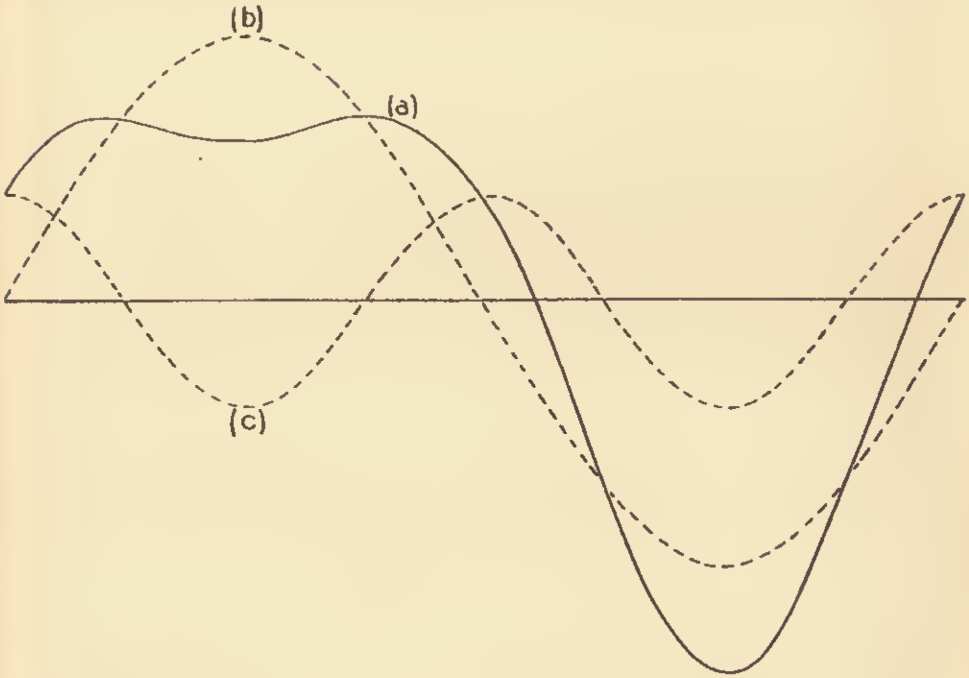

FIG. 26.1. Illustration of a particular type of double high water, showing partial tides.

26.3. The part played by shallow-water tides

In order to reveal the character of the phenomena, let the curve (a) of Fig. 26.1, illustrating the double high waters, be represented as closely as possible by a simple harmonic curve, by sketching, as (b) in Fig. 26.1. This must be drawn so that it is symmetrically disposed in relation to the high waters and to the low waters and also to mean sea level. Now take the differences of the ordinates of the two curves, as shown by the line (c). It is at once apparent that this represents a quarter-diurnal oscillation, so that this particular curve of double high waters can be expressed by the combination of a semidiurnal curve and a quarter-diurnal curve. It is clear that an essential feature of *this example* of the phenomenon is the occurrence of the quarter-diurnal tide. We shall show later that other species of shallow-water tides may be responsible for double high waters or low waters, but for the present we shall continue to consider only the quarter-diurnal species.

We have shown in Chapter VIII (see also Chapter XVII) that a wave cannot progress in shallow water without undergoing distortion, and that this distortion is expressible by quarter-diurnal constituents and also by higher species of constituents such as the sixth and eighth-diurnals. We also showed in Art. 8.2 that in the case of a progressive wave, if the primary wave can be represented by A cos $(nt - g)$, or

by A cos nt if we choose a time origin so that $g = 0$, then the secondary wave (or quarter-diurnal wave) can be represented by

$$B \cos (2nt + 90^\circ)$$

We showed also in Art 8.2 that the effect of shallow water on a progressive wave is to make the tide rise more quickly than it falls. However far the wave travels the result is only a cumulative distortion of this character, and it never leads to double high waters. But the question may arise from the exponents of the theories above discussed as to whether two such waves from different sources can interact so as to give double high waters. Let the two waves arrive at times $-$ T and $+$ T so that they are represented by

$$A \cos (nt + nT) + B \cos (2nt + 2nT + 90^\circ)$$

and

$$A \cos (nt - nT) + B \cos (2nt - 2nT + 90^\circ)$$

The combination of these two expressions gives :

$$(2A \cos nT) \cos nt + (2B \cos 2nT) \cos (2nt + 90^\circ)$$

It is seen at once that the form of this expression is exactly the same as the form of the expressions for the two component waves. Hence two progressive waves distorted by shallow water will only combine to give a wave distorted in the same manner ; under no circumstances will they yield the phenomenon of double high waters. The expression just obtained can be considered with various values of T ; thus it is possible for the amplitude of either part to be zero according to whether cos nT or cos $2n$T is zero, but it can readily be seen by graphical methods that so long as the phase of the second part is equal to 90° plus twice the phase of the first part, double high waters cannot occur.

Hence even the introduction of shallow water terms into the progressive waves does not justify the theories above discussed.

We shall proceed to discuss the conditions necessary to the generation of double high waters by shallow-water tides, firstly by the quarter-diurnal tides, and then by the higher species of shallow-water tides.

26.4. Conditions for generation by quarter-diurnal tides

On reference again to Fig. 26.1, we see that the quarter-diurnal wave has a trough at about high water of the semidiurnal wave, and a little consideration shows that this is really essential to the phenomenon. Hence we conclude that if the semidiurnal wave is approximately represented by

$$A \cos nt$$

then, for the occurrence of double high waters, tbe quarter-diurnal wave must be approximately represented by

$$B \cos (2nt + 180^\circ)$$

It is not essential that the phase of the latter should differ from twice the phase of the former by exactly 180°, but it is essential that it should be approximately 180°.

If we take a series of values of B relatively to A and combine the results we quickly see that the first effects of the quarter-diurnal part are to flatten out the high waters ; this is the explanation of the long " stand " of the tide which occurs in certain places. Then as B/A increases, the double high waters appear. Similar reasoning relative to the occurrence of double low waters shows that the phase of the quarter-diurnal wave must be approximately equal to twice the phase of the semidiurnal wave.

It is not sufficient, however, for the phase relationships of the semidiurnal and quarter-diurnal constituents to be correct—the amplitudes must also be considered.

It is easily shown that the amplitude of a quarter-diurnal tide must be at least equal to one-quarter of that of a semidiurnal tide to produce double high waters or

double low waters. For if we take the semidiurnal tide as $A \cos nt$ and the quarter diurnal tide as $-B \cos 2nt$, then for small values of t these are equal to

$$A\left(1-\frac{n^2t^2}{2}\right) \text{ and } -B\left(1-\frac{4n^2t^2}{2}\right)$$

according to the expression

$$\cos\theta = 1 - \tfrac{1}{2}\theta^2 + \ldots$$

if θ is small. These assert that the fall of level due to the semidiurnal tide in time t is $\frac{1}{2}An^2t^2$ and the rise due to the quarter-diurnal tide in the same time is $2Bn^2t^2$. Unless, therefore, this rise exceeds the fall due to the semidiurnal tide, there cannot be double high waters. The condition required is clearly that

$$4B > A \qquad \text{(26.4a)}$$

The theory for double low waters is very similar, and leads to the same result.

26.5. Conditions for generation by higher species of shallow-water tides

Previous articles show that double high waters are essentially due to superposing upon the main tide another tide of shorter period. We have already considered the simplest type of generation by a quarter-diurnal tide, and we must consider the possibilities of generation by a sixth-diurnal tide.

It is obvious that we must have the same conditions of phase-relationships as in the previous case; that is, the low water of the sixth-diurnal tide must occur at or about the same time as the high water of the semidiurnal tide in order to produce double high waters.

The quantitative relations between the amplitudes require the sixth-diurnal amplitude to be only one-ninth of the semidiurnal amplitude, as may be proved in the same manner as in the quarter-diurnal case. Thus if we take the semidiurnal tide as $A \cos nt$ and the sixth-diurnal tide as $-C \cos 3nt$, then for small values of t these are equal to

$$A\left(1-\frac{n^2t^2}{2}\right) \text{ and } -C\left(1-\frac{9n^2t^2}{2}\right)$$

and the condition for double high waters is

$$9C > A \qquad \text{(26.5a)}$$

This mode of generation differs from that of generation by quarter-diurnal tides in that the sixth-diurnal tide affects high waters and low waters alike, whereas the quarter-diurnal tide, if it gives double high waters, will accentuate the low water. This is due, of course, to the speeds of the tides being such that if the quarter-diurnal and sixth-diurnal tides are 180° out of phase with the semidiurnal tide at high water, then the quarter-diurnal tide will have the same phase as the semidiurnal tide at low water, whereas the sixth-diurnal tide will still be 180° out of phase.

In other words, if the quarter-diurnal tide did not exist, then double high waters due to the sixth-diurnal tide would be accompanied by double low waters as well, whereas quarter-diurnal tides may give either double high waters or double low waters but not both at the same place.

As the sixth-diurnal tides never exist without quarter-diurnal tides, it is evident that it is necessary to consider both together. As a simple case, consider the three partial tides combining to give

$$A\cos nt - B\cos 2nt - C\cos 3nt$$

The phase relations are such that both the quarter-diurnal tides and the sixth-diurnal tides tend to the generation of double high waters. As before, we can express the resultant near high water by

$$(A - B - C) + \tfrac{1}{2}n^2t^2(-A + 4B + 9C)$$

so that the condition for double high waters is

$$(4B + 9C) > A \quad . \quad . \quad . \quad . \quad . \quad (26.5b)$$

At low water, with a change in time origin, the tides would be represented by

$$- A \cos nt - B \cos 2nt + C \cos 3nt$$

and this can be expressed near low water by

$$(- A - B + C) + \tfrac{1}{2} n^2t^2 (A + 4B - 9C$$

The condition for double low waters is then

$$(9C - 4B) > A \quad . \quad . \quad . \quad . \quad . \quad (26.5c)$$

It is evident that the relative values of A, B, C, may be such as to satisfy (26.5b) and not (26.5c) and therefore the quarter-diurnal tide may assist the sixth-diurnal tide to produce double high wat rs and may resist the development of double low waters as well.

It is also evident that it is possible to have values of B and C so that neither the quarter-diurnal nor the sixth-diurnal tide alone would produce the double high waters.

It would be tedious to consider other possible combinations of shallow-water tides, and we shall not attempt to do so, but we can deduce a general principle from the above results. If the quarter-diurnal tide has a suitable phase relationship with the semidiurnal tide but has not a sufficiently large amplitude actually to produce double high waters it will tend to produce a marked flattening out of the high water curve and in the limiting case there will be a pronounced stand of tide. Now it is evident that if the curve is much flattened out over an interval of time comparable with, say, half the period of the sixth-diurnal tide, then this latter tide will be revealed at once by its oscillations about the approximately steady state on which it is superposed. Further, any short-period tide will be readily revealed under these conditions.

Hence double high waters may not be due to any single species of shallow-water tide but to the combined effects of several species. What is essential to the phenomenon is firstly that *one* of the shallow-water tides (not necessarily even the quarter-diurnal tide) must have the right phase relationship and a sufficiently large amplitude to give a short stand and the next higher species will then tend to produce the double tide.

26.6. Relation to standing oscillations

We shall now examine the conditions under which we can have the proper phase relations and amplitude relations requisite to the formation of double high waters. For simplicity, we shall consider the matter in relation to the quarter-diurnal tides, but it must be understood that the remarks here made may be applicable to other species of shallow-water tides.

So far as the amplitude relations are concerned we have two possible ways in which the ratio of amplitudes of quarter-diurnal and semi-diurnal tides may increase, either by a large increase in the quarter-diurnal amplitude or by a large decrease in the semidiurnal amplitude. The former is not likely, for in general, double high waters are only found in a limited area, and as shallow-water tides must be generated by a slow and cumulative process we must look rather to a relative and local diminution in the amplitude of the semidiurnal tide. These conditions are exactly satisfied in the case of standing oscillations near a node, and further, we indicated in Art. 8.2 that if the phase-lag of M_2 is g, then the phase-lag of M_4 under these circumstances is either $2g$ or $2g + 180°$. In other words, the general theory of standing oscillations indicates that the shallow water terms of the quarter-diurnal species are related to the primary semidiurnal terms in exactly the manner that is required to produce double high waters or double low waters.

Further, the nodes of the quarter-diurnal oscillations will not occur at the same place as the nodes of the semidiurnal oscillations. This may be seen from the fact that if c is the rate of propagation of a wave and T its period, then the wave-length is cT, so that the wave-length for a quarter-diurnal oscillation is only half that of a semidiurnal oscillation. The profiles of two such waves in a channel closed at one end are therefore as depicted in Fig. 26.2, and we see that at the first node of the semidiurnal oscillation the amplitude of the quarter-diurnal oscillation as at its maximum; that is, the phase of the semidiurnal oscillation is changing quickly whilst that of the quarter-diurnal is stationary.

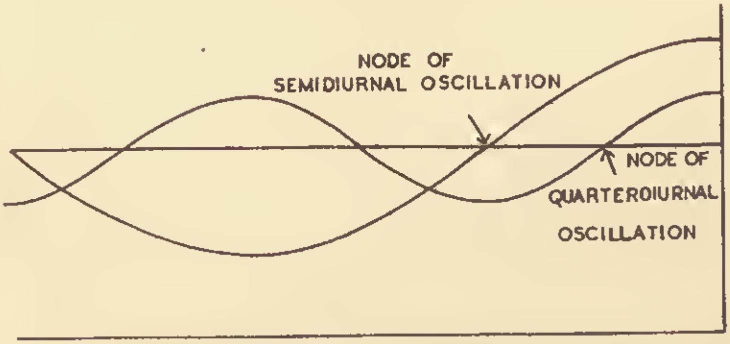

FIG. 26.2. Elevations of semidiurnal and quarter-diurnal standing oscillations in channel closed at one end.

26.7. Discussion of tides at Portland and Southampton

These theoretical conclusions can now be examined in relation to the tides at Portland and Southampton, using the harmonic constants :—

	Portland	Southampton	Freshwater bay
g of M_2	194°	329°	276°
g of M_4	030°	019°	016°

On these we make comments of interest as follows :—

(1) There is a rapid change in the phase of M_2 from Portland to Southampton. It was stated in Art. 25.4 that the tides in the English channel (in the absence of the earth's gyration) could be considered as standing oscillations with a node across the Channel from near the Isle of Wight. This nodal line would be modified by the effect of the earth's gyration to produce an amphidromic point, but in actuality, probably owing to friction, the amphidromic system is degenerate; that is, the actual point is on the land though the cotidal lines radiate from it. A glance at the chart of cotidal lines for the English channel (Fig. 24.10) will show how the cotidal lines radiate from a point north-west of the Isle of Wight. Under such circumstances the range of tide decreases towards the amphidromic point, and we have the conditions fulfilled, that the places at which double low waters and high waters occur are near a nodal line or amphidromic point.

(2) We see that though the phase of M_2 changes very quickly, the phase of M_4 changes very little. This also was predicted as a consequence of the theory of standing oscillations, as in Art. 26.6.

(3) At Portland we have

$$2(g \text{ of } M_2) - (g \text{ of } M_4) = -002°$$

One of the conditions specified for double low waters was that the phase of the quarter-diurnal tide should be nearly equal to twice the phase of the semidiurnal tide, and we see that at Portland this condition is fulfilled.

(4) At Freshwater bay, on the south-west coast of the Isle of Wight, we have

$$2(g \text{ of } M_2) - (g \text{ of } M_4) = 176°$$

This is the most favourable phase-relationship for the production of double high waters.

(5) When, however, we consider the tide at Southampton we have

$$(g \text{ of } M_4) = 2(g \text{ of } M_2) - 279°$$

The standing wave theory would indicate

$$(g \text{ of } M_4) = 2(g \text{ of } M_2) - 180°$$

while the progressive wave relationship would indicate

$$(g \text{ of } M_4) = 2(g \text{ of } M_2) - 090°$$

Hence we have the curious result that at Southampton the quarter-diurnal tide is not related to the semidiurnal tide by either of the standard relationships or approximations thereto. If we combine the curves for M_2 and M_4 we do not get double high waters; in fact, with the phase relationships above we should not expect them. Since it is evident that the phase-relationships between M_4 and M_2 for a progressive wave are completely reversed at Southampton, then we should expect that the tide will fall more quickly than it rises, which is the reverse of what is customary in progressive waves and in most estuaries.

With regard to the amplitudes we have

	Portland	Southampton	Freshwater bay
H of M_2	2·07 ft.	4·46 ft.	2·02 ft.
H of M_4	0·41 ft.	0·82 ft.	0·53 ft.
Ratio	0·20	0·18	0·26

and it is at once evident that even at Portland, where the phase-relationship between M_4 and M_2 is suitable for the production of double low waters, the amplitude of the quarter-diurnal tide is not, on the average, great enough to give more than a long stand at low water, though probably at spring tides the ratio of amplitudes will become great enough to produce double low waters. It is probable, therefore, that we must look to the sixth-diurnal tide and possibly higher species of shallow-water tides to give the observed double low waters at Portland.

At Freshwater bay we note that both conditions required for the production of double high waters in the simplest possible manner are fulfilled; viz. the phase-relationship is most favourable, as we have already noted, and the ratio of amplitudes satisfies the condition (26.4a). Hence, on the south-west coast of the Isle of Wight, the conditions are highly favourable to the production of double high waters in accordance with the theory of formation in the simplest possible manner by

(*a*) the diminution of the semidiurnal tide relatively to the quarter diurnal tide, and

(*b*) the occurrence of the correct phase-relationship of the semidiurnal and quarter-diurnal tides.

We shall have to inquire further into the mechanism of the phenomena at Southampton, but we can remark at this point that it is probable that it is the presence of the Isle of Wight with the two approaches to Southampton which is responsible for the failure to produce double high waters by the simplest possible means. It is certain that the Isle of Wight will not affect the quarter-diurnal oscillations in the main part of the English channel, since these are in a stable region for this type of oscillation; that is, they are not in the neighbourhood of a node or amphidromic point for tides of that species. The semidiurnal tides, however, are in such a region and are liable to change quickly so that the presence of two approaches to Southampton probably causes a perturbation of the semidiurnal tide.

26.8. Further investigation of the Southampton tides

This matter is of some importance, so we shall give here the results at Southampton of an investigation of a tide curve for a particular day. The actual curve has been read off in "hours" of one-twelfth the period of oscillation, and the readings are denoted by X in the following table. The curve has been analysed for the species of tides up to and including the eighth-diurnal, and the values of these are denoted by Y_2, Y_4, Y_6 and Y_8. Thus Y_2 is the first approximation to the tide, being the semidiurnal part. A better approximation is $Y_2 + Y_4$, a better one still is $Y_2 + Y_4 + Y_6$, and so on. In Fig. 26.3, we give the graphs for X, $Y_2 + Y_4$ and $Y_2 + Y_4 + Y_6$. We see that the contribution of the semidiurnal and quarter-diurnal tides gives us no suggestion of double high waters, but when the sixth-diurnal tides are included there is a pronounced flattening of the curve at high water, and the characteristic "halt" near the half-tide on the rising tide appears in the curve. The inclusion of the eighth-diurnal tides, if the curves are carefully drawn, tends to produce double high water, but just fails to do so.

If we take the difference between the original observations (X) and the synthesised curve ($Y_2 + Y_4 + Y_6 + Y_8$) we get the residues Z, and we see that these have, at least, five complete periods. Hence the tenth diurnal is here indicated, and it is of appreciable magnitude (about 0·2 ft.). If this curve were further examined, the higher species of tide would be indicated.

TABLE 26.1

Analysis of Tidal Curve for Southampton

Hour	X	Components				Residue
		Y_2	Y_4	Y_6	Y_8	Z
0	− 0·7	0·71	− 0·70	− 0·92	− 0·02	0·2
1	2·4	3·78	− 1·73	0·87	− 0·19	− 0·3
2	6·3	5·85	− 1·03	0·92	0·21	0·3
3	5·8	6·34	0·70	− 0·87	− 0·02	− 0·3
4	6·0	5·13	1·73	− 0·92	− 0·19	0·3
5	4·5	2·56	1·03	0·87	0·21	− 0·2
6	− 0·5	− 0·71	− 0·70	0·92	− 0·02	0·0
7	− 6·5	− 3·78	− 1·73	− 0·87	− 0·19	0·1
8	− 7·7	− 5·85	− 1·03	− 0·92	0·21	− 0·1
9	− 4·7	− 6·34	0·70	0·87	− 0·02	0·1
10	− 2·7	− 5·13	1·73	0·92	− 0·19	0·0
11	− 2·3	− 2·56	1·03	− 0·87	0·21	− 0·1

From this investigation of the tides at Southampton, we conclude that no simple explanation can be given as to the mode of production of the double high waters at that place.

In general, we have shown that the production of double high waters depends upon the occurrence of favourable amplitude-ratios and phase-relationships between the semidiurnal tides and the shallow-water tides. The conditions for the production of double high waters are as follows, in order :—

(1) the amplitude-ratios can only become large enough by the diminution of the semidiurnal tide, so that as a rule no double high waters can occur at a place unless it is near a node or amphidromic point (real or degenerate) for the semidiurnal tide ;
(2) if the phenomenon is to be produced in the simplest possible manner the amplitude-ratio and phase-relationship between the semidiurnal tide and the quarter-diurnal tide must be favourable ;
(3) failing this, the sixth-diurnal tide must have favourable relations with the semidiurnal tide ;
(4) failing this again, the higher species of shallow-water tides must have the proper relations with the semidiurnal tide.

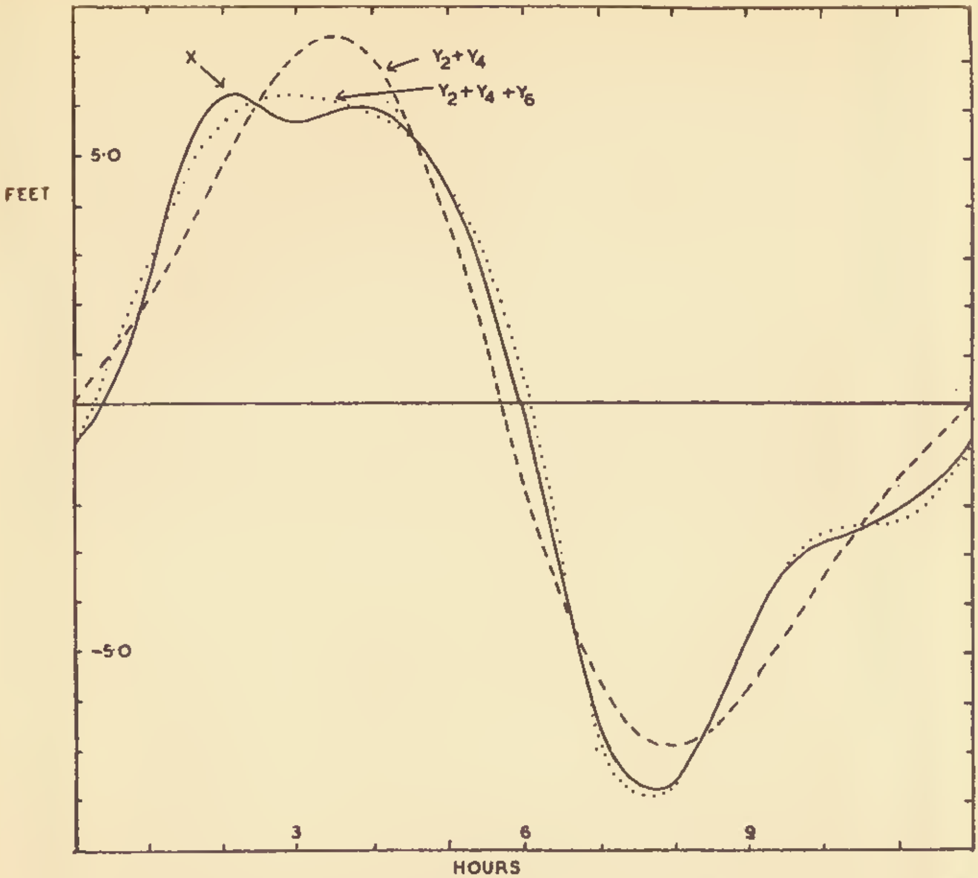

Fig. 26.3. Tide curve at Southampton, with successive approximations by summing partial tides.

We have shown that condition (1) is approximately fulfilled at Southampton but that condition (2) is not even approximately fulfilled, so that the more complicated methods of producing double high waters have to be considered. We find that the sixth-diurnal tide alone is not capable of producing the double high waters, but it contributes largely to the phenomenon in that it tends to give a long stand at high water. The higher species of shallow-water tides then conspire to give the double high water.

It may also be noted, in confirmation of what has been repeatedly set forth, that at places like Southampton, the prediction of tides by direct harmonic methods is practically impossible, owing to the large number of species of shallow-water constituents that would be required.

CHAPTER XXVII

BORES, OVERFALLS AND RACES

27.1. General description of bores

In most parts of the world the tides change slowly and continuously throughout the cycle of the phenomena, but in certain regions there are discontinuities in the motion which produce very striking effects. The tide may rise suddenly and very rapidly, so that a wall of water, so to speak, rushes up a channel and menaces with death or destruction all that lies in its path. Readers of Sir Walter Scott's novel "Redgauntlet" will remember the effective use he makes of this phenomenon when he writes of the alarm of his hero as he recollected that in the upper part of the Solway firth, "the tide advances with such rapidity upon the fatal sands that well-mounted horsemen lay aside hopes of safety if they see its white surge advancing while they are yet at a distance from the shore." In a later episode he writes of "the foaming crests of the devouring waves, as they advanced with the speed and fury of a pack of hungry wolves."

Such a sudden inrush of water is generally known as a bore or eagre, and the principal characteristic of a bore is the relatively quick rise in level. It is only very rarely that a bore has a vertical front like a step, and in all cases the vertical front is only a small fraction of the total elevation of the bore. The main part of the profile of the bore, as seen from the banks of the channel, consists of a steep slope and this again is furrowed by large waves, more or less permanent. In many cases where the vertical front does not occur the rapid rise of water level is prominently associated with the passage of these characteristic waves. A bore can travel along a water surface as in a river or along the sands of an estuary, the principal characteristic, as has been said before, being that of the abnormal rapidity of change of water level. In the Solway firth the water may rush forward as a huge wave 3 to 6 feet high, which is resolved into a great mass of water in violent perturbation. In the Severn the bore rushes up the river at the rate of about 10 to 20 miles per hour, with a height of 3 to 7 feet.

Large bores occur in other parts of the world, but the most striking example is that of the Chien tang kiang, in China; it was studied by the late Admiral W. Usborne Moore in 1888 and 1892, when commanding H.M. Survey ships *Rambler* and *Penguin*. Some of the boats on one occasion were anchored in what was understood to be a safe position; but just after the bore had passed well clear of the boats they were struck by a violent rush of water in a succession of big waves; in 10 minutes the water rose 9 feet and in spite of the engines the 8-knot current dragged the anchors, and the boats were carried for 3 miles. Incidentally, when hove up, the boats' chain cables were as bright as polished silver Admiral Moore describes the bore as a "cascade of bubbling foam" falling forward at an angle of between 40° and 70°, with a height of 8 to 11 ft. (though 15 ft. may be attained); the bore passes Haining with a roar but little inferior to the rapids below Niagara, and about $1\frac{3}{4}$ million tons of water pass in one minute. He relates amusing stories of the reactions of the Chinese to the phenomenon; also how they make profitable use of it. At various points along the river they make sheltered platforms, and the junks wait here until the bore rushes past; a rampart is used to deflect the main wave of the bore so as to preserve the junks from danger. After the bore has passed, the water rises rapidly on the platform, the junks float in safety, and they take advantage of the after-rush so as to proceed up the river under natural power at the rate of about 10 knots.

There is evidence to show that at one time this bore did not exist, and experience has shown that artificial changes in a river have destroyed bores that once existed.

Before the Seine was improved (commencing about 1780) it was subject to a very dangerous bore which affected the whole river to Rouen. The bore now only occurs at the greatest spring tides and affects only a short length of the river. All the evidence points to a bore being due to a nice balance between various conflicting forces ; natural changes, by silting or otherwise, in an estuary will make all the difference between the existence and non-existence of a bore.

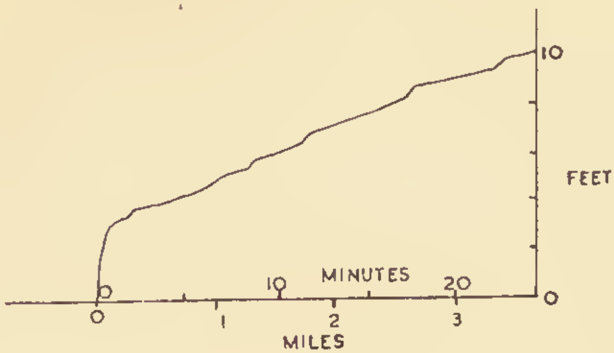

FIG. 27.1. Profile of bore in Petitcodiac river.

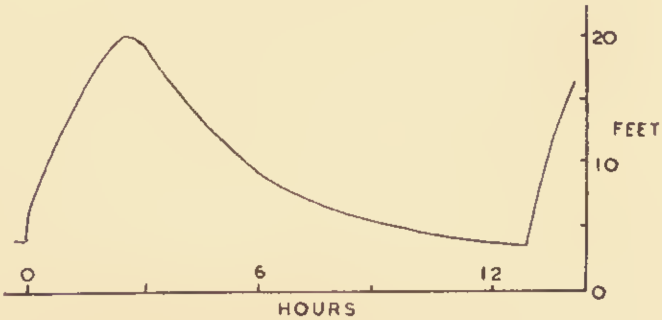

FIG. 27.2. Shape of tide curve in River Trent.

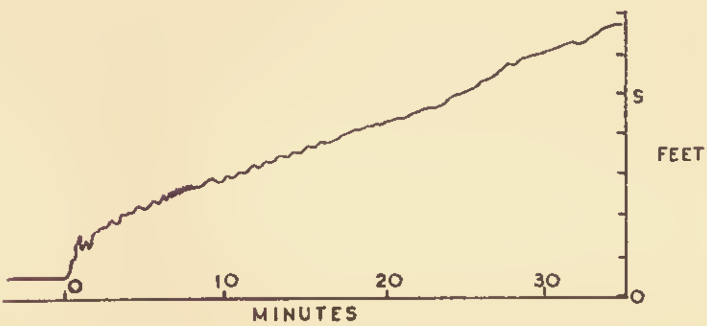

FIG. 27.3. Profile of bore in River Trent.

The shape of the surface of the water in the bore has been determined for several places. Fig. 27.1 shows the profile of the bore in the Petitcodiac river from observations by Dr. W. Bell Dawson, and Figs. 27.2 and 27.3 show the shape of the tide curve and the profile of the bore in the River Trent, as deduced by Champion and Corkan. These show the vertical front characteristic of the bore, as is seen in the photograph of Fig. 27.5, for the River Trent.

FIG. 27.4. Photograph of bore in Petitcodiac river.

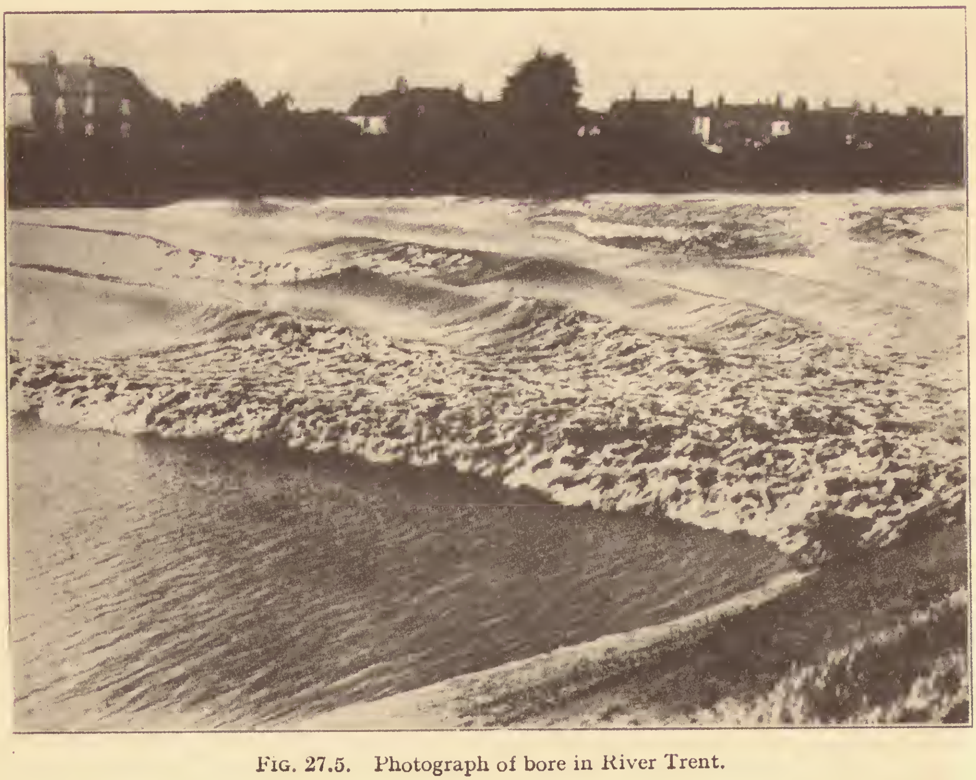

FIG. 27.5. Photograph of bore in River Trent.

[*To face p.* 229.

Curves of rise of tide in a more open scale also show characteristic waves on the surface of the bore, as is apparent in Fig. 27.3 and the photograph of Fig. 27.5 for the Trent.

27.2. The bore in the Trent

A very elaborate investigation of the bore in the Trent was commenced by the late H. N. Champion in 1928, and the observations were reduced by R. H. Corkan, of the Liverpool Observatory and Tidal Institute. The discussion of the observations revealed many significant features of the phenomenon. The Trent flows into the Humber and the profile of the river bed is indicated in Fig. 27.6 by the curve A B C.

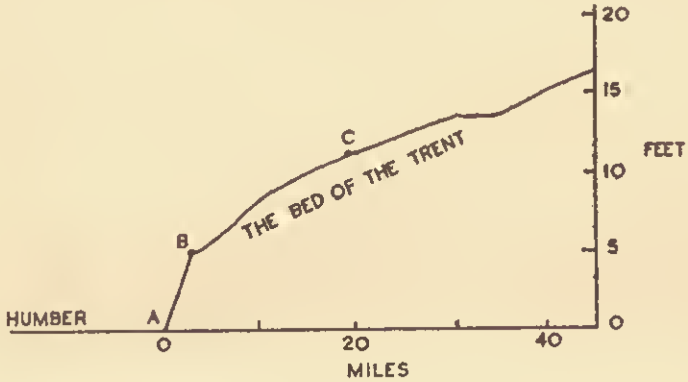

FIG. 27.6. The gradients in the bed of the River Trent.

At the point A the Trent enters the Humber, and from A to B (a place called Burton Stather) the river bed rises sharply. There is no bore in this region. From B to C (at Walkerith) the gradient of the bed of the river is much less steep and the bore is a maximum at the point C. After this point the gradient again diminishes and the bore also diminishes.

It is also shown by Corkan that the rate of travel of the bore steadily increases from B to C and thereafter increases rapidly.

It appears, therefore, that one of the essential features to be considered is that of the steepness of the gradient of the river bed, and probably another factor arises from restrictions in the channel. We shall proceed to discuss the latter subjects.

27.3. The effects of restrictions in a channel

Very great caution is required before making assertions as to the effects of restrictions in a channel. Take, for instance, a channel in which a steady current is flowing. If at some point the channel is narrowed it is obvious that the velocity of the current must be increased in order to convey the same amount of water in the same interval of time as in the wider part of the channel. A point that is by no means easy to settle is whether a mere change of velocity is or is not likely to give equal rates of transport of volume as well as equal rates of transport of energy. Some adjustment of elevation of the water surface will be required and the necessary mathematical analysis will be given in the next article, for the case of a current flowing in a channel which is suddenly narrowed at a certain point. The channel is supposed to be of unlimited length and the bed of the channel to be a level plane. Then the mathematical analysis of Art. 27.4 yields the result:

$$\frac{\text{change in elevation}}{\text{change in breadth}} = \frac{u^2/gb}{1 - u^2/gh} \qquad . \quad . \quad . \quad (27.3a)$$

where b is the mean breadth of the broad and narrow parts of the channel
h is the mean depth of the fluid in the two portions
and u is the mean velocity of current.

This result is of great theoretical and practical importance, for we see that whether the elevation of the surface increases or decreases as the breadth decreases depends on whether u^2 is greater or less than gh. It will be remembered that the mean rate of propagation of a free progressive wave is equal to $\sqrt{gh}$, as shown in Art. 17.8, so that we have this special velocity appearing again.

If, therefore, the current has a mean velocity which is greater than the rate of propagation of a progressive wave a constriction in the channel will cause the elevation to be increased. Otherwise the elevation will fall.

It is a simple deduction that if the channel, instead of having parallel plane walls, has one side plane and the other corrugated, then the elevation will have a

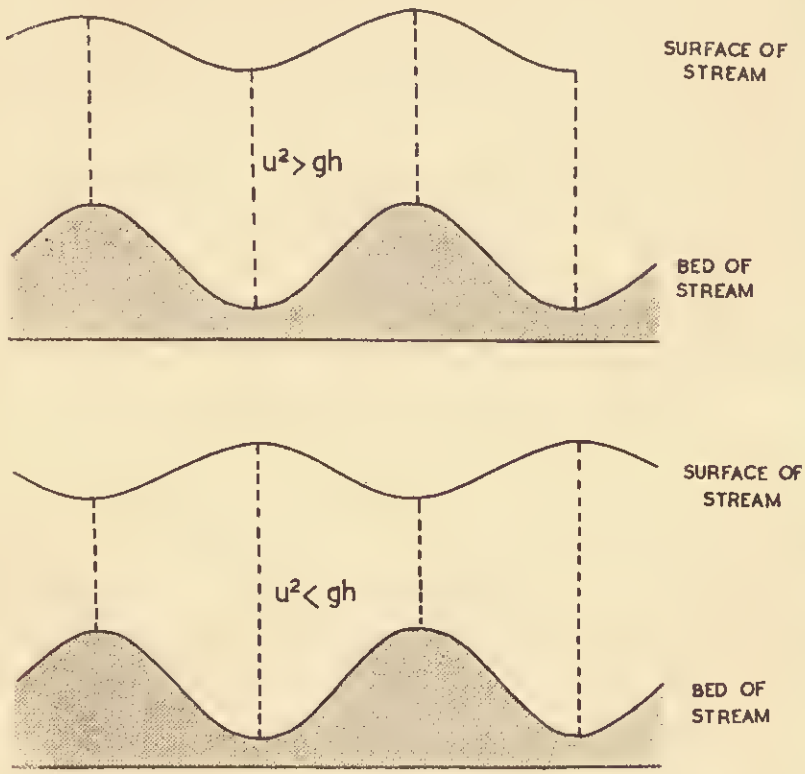

FIG. 27.7. Effect of corrugations in bed of channel.

shape similar to that of the corrugated wall but on a different scale and having peaks where the passage is narrow and troughs where the passage is broad, provided that the rate of current exceeds the rate of propagation of a free progressive wave in water of the same depth. If, however, the current has a smaller velocity than this critical value then the profile will be inverted, and the peaks will correspond with the wide portions of the channel.

An exactly similar result follows from a discussion of the effect of corrugations in the bed of the stream, the channel having parallel plane walls. The results are shown diagramatically in Fig. 27.7.

Similar diagrams, of course, can be drawn for the corrugated walls The corrugations can be regular or irregular.

27.4. Mathematical investigation of the effects of restrictions in a channel

Consider a channel of rectangular cross section and variable breadth and depth. It will be convenient to consider two sections and to measure the depths and the

surface elevations from the same mean elevation, so that in one section we take depth $h + h'$ and elevation y and in the other section we take depth $h - h'$ and elevation $-y$. Let the corresponding breadths of channel be denoted by $b + b'$ and $b - b'$, and let the corresponding velocities of current across the two sections be denoted by $u + u'$ and $u - u'$. The mean values of breadth, depth, and velocity are thus b, h, u. Then, as the same quantity of fluid must be transported across each sectional plane in the same time, we must have

$$(b + b')(h + h' + y)(u + u') = (b - b')(h - h' - y)(u - u') \quad . \quad . \quad . \quad (27.4a)$$

whence, if we neglect products of the small quantities b', h', u' and y, we get

$$b'hu + h'bu + ybu + u'bh = 0$$

and therefore

$$\frac{u'}{u} = -\frac{b'h + h'b + yb}{bh} \quad . \quad . \quad . \quad . \quad (27.4b)$$

Also, Bernoulli's equation (17.9r) for a stream line at the surface gives

$$gy + \tfrac{1}{2}(u + u')^2 = -gy + \tfrac{1}{2}(u - u')^2 \quad . \quad . \quad . \quad (27.4c)$$

or

$$gy + uu' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (27.4d)$$

On substituting for u', we get

$$y(gh - u^2) = \frac{(b'h + h'b)u^2}{b} \quad . \quad . \quad . \quad . \quad (27.4e)$$

Alternatively, substituting for y, we get

$$u'(gh - u^2) = -\frac{(b'h + h'b)gu}{b} \quad . \quad . \quad . \quad (27.4f)$$

We shall consider the application of these two equations to two special cases:

(1) A channel of constant depth ($h' = 0$) and variable breadth.

In this case, if we take b' positive and $u^2 < gh$, then y is positive and u' is negative. The interpretation is that if we pass from a broad portion to a narrow portion of the channel, the surface elevation is decreased while the velocity of current is increased.

If, however, b' is positive and $u^2 > gh$, then if we pass from a broad portion of the channel to a narrow portion, the surface elevation is increased and the velocity of current is decreased.

(2) A channel of constant breadth ($b' = 0$) and variable depth.

In this case, if we take h' positive and $u^2 < gh$, then y is positive and u' is negative. The interpretation is that if the depth diminishes, then the surface elevation is decreased and the velocity of current is increased. If, however, $u^2 > gh$, then the surface elevation is increased and the velocity of current is decreased.

27.5. Application to overfalls and races

An immediate result of the theory of Art. 27.4 is the application to the problems of water flowing over broken ground or over submarine precipices. The profile of the broken ground tends to be reproduced, directly or indirectly, at the surface, and a submarine precipice will cause a rapid change in the velocity of the stream.

If the depth is very much altered then a critical state may be approached in which u^2 is actually equal to gh. On the basis of the formula (27.4e), we see that this would involve an infinite change in the elevation, but in actuality, of course, we should not get such an infinite value, as frictional forces, which have been neglected systematically, would then seriously affect the motions. But it is clear that when this critical value is approached then comparatively small changes in the depth may so affect the rate of current or stream that the motion is unstable. Thus a "race" or excessively broken and dangerous water may be caused by changes in the sea-bed which themselves would not be expected to cause dangers to small vessels.

In ordinary tidal phenomena the rate of the stream is very much less than that of the rate of propagation of a free wave. In practically all cases, therefore, the effects of irregularities in the sea bed will be *inverted*, according to the foregoing theory, so that a ridge in the sea bottom will cause a trough in the sea surface.

To the ordinary reader, this seems a paradox. He would expect a ridge in the sea bottom to produce a ridge in the sea surface. If we return to the fundamental concepts of

(1) transmission of volume
(2) transmission of energy

we see that relative variations in h and u according to (1) are equal and opposite, but the paradox arises from the fact that the energy varies as the square of the velocity of the current, and it is this which dominates the situation.

*27.6. Motion in an inclined channel

The theory of motion in an inclined channel is also required in connection with bores.

Let distance OX in a horizontal plane be denoted by x, let the bed OC of a rectangular channel make an inclination i to the horizontal (Fig. 27.8), and let the

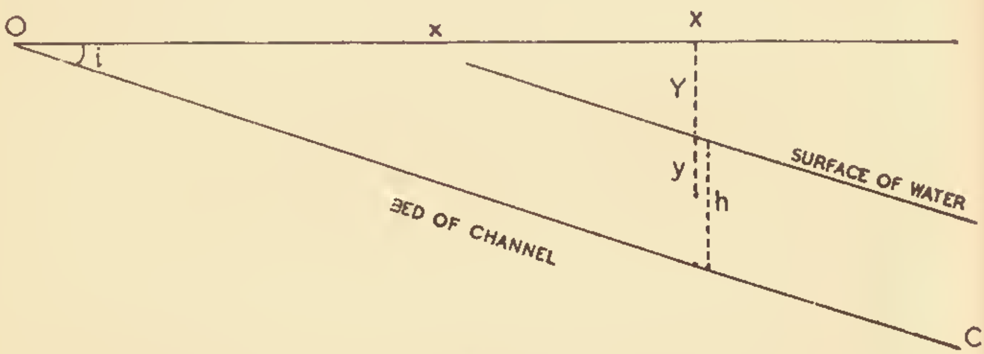

FIG. 27.8. Stream in inclined channel.

surface of the water below the horizontal plane be denoted by Y. Then if distance below the surface of the water is denoted by y, we have Bernoulli's equation (17.9r) yielding, for a particular stream tube,

$$H = -(y + Y)$$

$$\frac{p}{\rho} + \tfrac{1}{2}u^2 - g(y + Y) = \text{constant}$$

where p, the pressure, is equal to $\rho g y$, so that we get

$$\tfrac{1}{2}u^2 - gY = \text{a constant} \quad . \quad . \quad . \quad . \quad (27.6a)$$

Also, if i be small
$$Y + h = xi \quad . \quad . \quad . \quad . \quad . \quad (27.6b)$$

Hence
$$\tfrac{1}{2}u^2 + gh = \text{constant} + xig$$

Take two near places X_1, X_2 and let suffixes 1 and 2 be used to denote variables pertaining to such places. Then we have

$$\tfrac{1}{2}(u^2_2 - u^2_1) + g(h_2 - h_1) = (x_2 - x_1)\, ig \quad . \quad . \quad . \quad (27.6c)$$

Let u be the mean value of u_1 and u_2, and let h be the mean value of h_1 and h_2, so that we take

$$u = \tfrac{1}{2}(u_1 + u_2), \quad h = \tfrac{1}{2}(h_1 + h_2)$$

and very approximately also
$$h_1 h_2 = h^2$$

* See par. 1, page vii.

We also have from the condition of transmission of volume

$$u_1h_1 = u_2h_2 = uh$$

Then we get

$$\begin{aligned}\tfrac{1}{2}(u^2_2 - u^2_1) &= \tfrac{1}{2}(u_2 - u_1)\,(u_2 + u_1)\\ &= u(u_2 - u_1)\\ &= u^2h\left(\frac{1}{h_2} - \frac{1}{h_1}\right)\\ &= -\frac{u^2h}{h_1h_2}(h_2 - h_1)\\ &= -\frac{u^2}{h}(h_2 - h_1) \qquad (27.6d)\end{aligned}$$

Hence we get, from (27.6c)

$$(h_2 - h_1)\left(g - \frac{u^2}{h}\right) = (x_2 - x_1)ig$$

or

$$\frac{(h_2 - h_1)}{(x_2 - x_1)} = \frac{i}{1 - u^2/gh} \qquad (27.6e)$$

The interpretation of this equation is that if u is less than the rate of propagation of a free wave, h increases with x, but if u is greater than the critical value $\sqrt{gh}$ the depth of water from the surface to the bed of the stream diminishes with x.

It is important to notice that this equation involves the square of the velocity of the current, so that the result is applicable to a current flowing downhill by ordinary natural causes or to a flow of water uphill as maintained by an increasing head of water at some point down the slope.

27.7. Generation of bores.

We are now in a position to examine the causes of formation of bores. A theory which is sometimes expounded is that of the distortion of wave-profile as a free progressive wave travels in shallow water. It is supposed that as the wave travels the front of the wave gets progressively steeper until it becomes vertical and falls over in the same way as a breaking wave on the sea shore. The weakness of this theory in its application to tides is that it supposes the shallow-water effect to be excessively great. Any such distortion would be cumulative and would become apparent over a very large area, whereas the tidal bore in the Trent, for instance, is manifestly a very local phenomenon not experienced even in the Humber.

The first consideration to be taken into account is that of the effects of an inclined channel. If the flow of water is extremely slow then the gradient of the surface is zero ;. that is, the water finds a normal horizontal level irrespective of the inclined bed, as in Fig. 27.9. In fact, if u is very small, we have the conditions almost of a lake where the inequalities of the bed are not apparent on the surface.

If the inclination of the bed of the stream to the horizontal is denoted by i then the gradient of the water surface relative to the bed of the stream is given by

$$\left.\begin{array}{l}\text{gradient of surface}\\ \text{relative to bed of channel}\end{array}\right\} = i \quad (u = 0) \qquad (27.7a)$$

When, however, the current is not zero, then the mathematical analysis of Art. 27.6 shows that

$$\left.\begin{array}{l}\text{gradient of surface}\\ \text{relative to bed of channel}\end{array}\right\} = \frac{i}{1 - u^2/gh} \qquad (27.7b)$$

We see, as in the case of restrictions in a channel (Art. 27.3) that the relation between the velocity of the current (u) and the rate of propagation of a free

progressive wave ($\sqrt{gh}$) is a critical one. The gradient of the surface is always greater than i, but when u^2 approaches the value gh then the gradient may become very great, according to the simple theory just utilised.

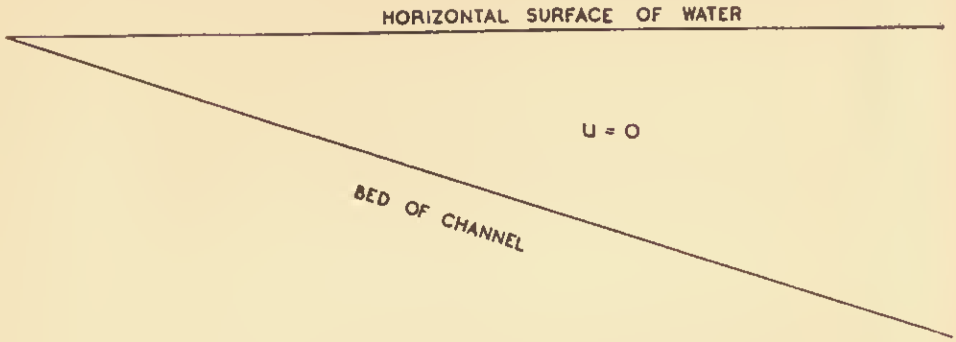

FIG. 27.9. Surface of slowly moving water in inclined channel.

Here, then, we have something which may be very local in its effects, as distinct from the theory referred to at the beginning of this article. Suppose, as in the case of the Trent, that the tide rises up a steep slope. Then as the depth rapidly diminishes a slope may be reached in which we have the critical relation

$$u^2 = gh,$$

as indicated in the formula (27.7b). The result of this is that for a very small change in distance the change in elevation would become very great, and would *tend* to give such a steep rise of front to the water surface as to approximate to a wall of water. We emphasise the word *tend* as other forces would come into operation. We note that the conditions of Fig. 27.10 would hold, as at first u^2 would be less than gh as in

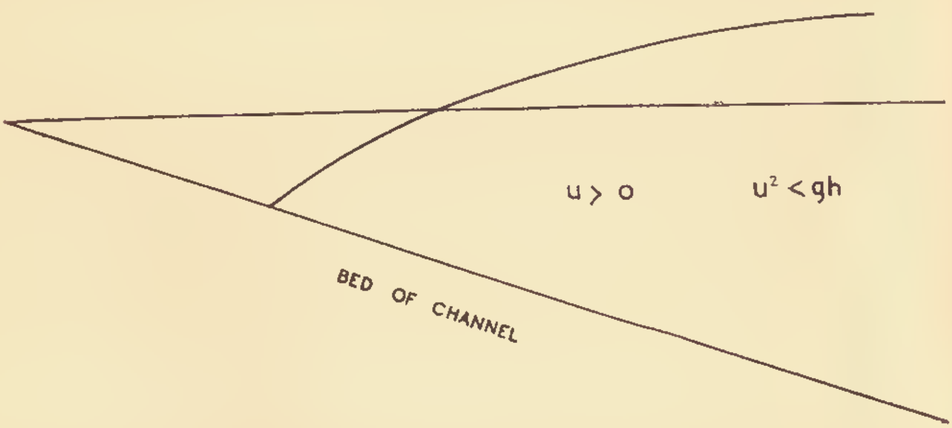

FIG. 27.10. Surface of stream flowing up an inclined channel.

normal tidal motion and would be forced up to a critical value. The steepness of the slope would not greatly affect the tendency to such a discontinuity in the profile of water surface, but according to (27.7b) the greater the inclination of the bed of the channel the greater the change in elevation. Hence a steep rise definitely favours the formation of a bore in two ways, firstly because of the general magnification of the slope of the water surface, and secondly by accelerating the rate at which the velocity of the water tends to approach the critical value $\sqrt{gh}$. (See Fig. 27.11.)

Again, reverting to Art. 27.3, we see that a restriction in the channel, whether in the walls or in the bed, would tend to yield the same result. We saw that when

u is less than the critical value the effect of a restriction in the channel is to diminish the elevation inside the narrow channel, and if u tends to the critical value $\sqrt{gh}$ this effect is greatly exaggerated. Hence a steep rise and a narrow entrance to a channel tend to give the same result, and if the conditions are favourable as regards depth so that the critical relations can be quickly obtained, a discontinuity in profile will result, and a bore will be formed.

It must be realised, however, that as such critical conditions are approached the conditions on which the formulæ have been obtained are largely violated so that the formulæ cannot be used with great precision. They can only indicate the possibilities. As the critical value is approached, instability sets in, and the wave motion associated with the bore follows as a necessity of this. The theory of this wave motion also can be investigated but it is somewhat complicated.

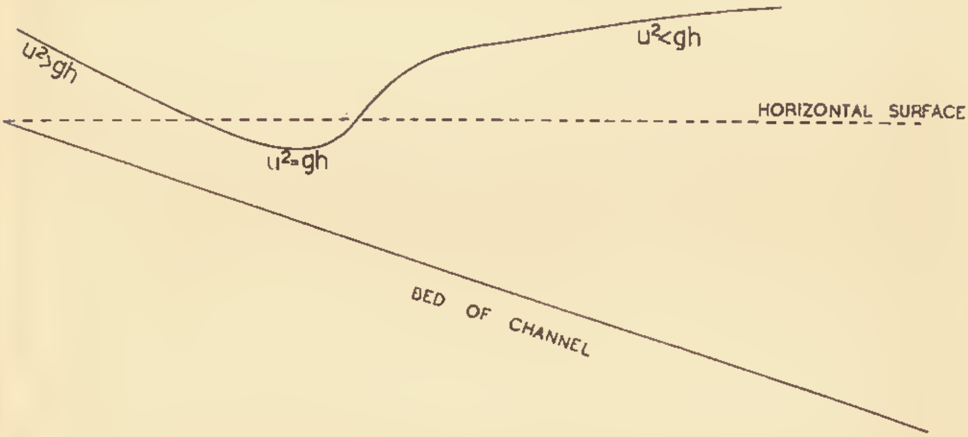

FIG. 27.11. Surface of stream in inclined channel when velocity becomes "critical."

In interpreting Fig. 27.11 we may consider it as a *stationary bore,* in two ways. Firstly, the current may be flowing down and by artificial restrictions we may get $u^2 > gh$ and then $u^2 = gh$. The theory asserts that the depth of water above the bed of the stream will thereafter increase. Alternatively, we may consider that the pressure of water on the right, however maintained, causes a current up the slope. In effect, water is being supplied on the right and disposed of on the left. We have not discussed how this would be effected in any ordinary channel.

If we consider the analogous problem of the last stage of a breaking wave on a sloping beach we see that Fig. 27.11 conforms to the familiar phenomenon, for as the wave breaks there is a steep front to it, and the broken water is carried rapidly up the beach as a thin stratum of moving fluid. In tidal motion shallow-water effects never reach this stage, though waves on a beach may do so, but not only because of a shallow-water effect, since we are dealing with the motion of water up a slope.

27.8. The propagation of a bore

The questions now arise as to how a bore is maintained and the rate at which it moves. If the tide in an estuary continues to rise, then a bore is forced onward and it will also tend to travel as a free wave. The problem of a bore invading still water is somewhat different from that considered in connection with the generation of a bore, and it is necessary to reduce it to a consideration of steady motion. This is effected as in Art. 17.10. If we imagine ourselves to be travelling with the bore at a rate denoted by c, then the still water appears to us to have a velocity equal to $-c$, and the velocity of current in the original direction of travel of the bore is $(u - c)$. We must replace u in previous articles by $(u - c)$, so that the condition for the bore then becomes

$$(u - c)^2 = gh \quad . \quad . \quad . \quad . \quad . \quad (27.8a)$$

and the mathematical theory given in the next article shows that the rate of propagation depends on the height of the bore (B) and the mean depth (h) before and after the bore, so that we get

$$c = \left(1 + \frac{1}{2}\frac{B}{h}\right)\sqrt{gh} \quad . \quad . \quad . \quad . \quad (27.8b)$$

as the rate at which the bore travels.

If, however, the bore travels against the current (U) in a river, then the rate of travel becomes

$$c = \left(1 + \frac{1}{2}\frac{B}{h}\right)\sqrt{gh} - U \quad . \quad . \quad . \quad . \quad (27.8c)$$

We see, therefore, that the bore travels forward at a rate considerably greater than that of a progressive wave ($\sqrt{gh}$), and that the rate of travel is diminished if the velocity of the river current increases.

Precise data are not available to illustrate these results for an actual bore, but Wheeler, in his " Tides and Waves," gives some very rough observations of the bore in the Trent. He states the rate of travel of the bore as 15 statute miles per hour (*i.e.*, 22 ft. per second) and the velocity of current immediately after the passage of the bore as 5 miles per hour (*i.e.*, 7 ft. per second). He estimates the depth of the river before and after the passage of the bore as 6 ft. and 12 ft. respectively. Since the mean depth is 9 ft., and $g = 32$ ft. per second per second, then $\sqrt{gh} = \sqrt{288}$ ft. per second, whence formula (27.8b) gives

$$c = \text{rate of travel of bore} = \frac{12}{9}\sqrt{288} \text{ ft. per second} = 23 \text{ ft. per second}$$

and formula (27.8a) gives

$$c - u = \sqrt{288} \text{ ft. per second} = 17 \text{ ft. per second}$$

The value of u is thus 6 ft. per second. The value for c is quite close to Wheeler's value of 22 ft. per second, and the mean current as given by Wheeler is 3·5 ft. per second if the bore invades still water. If we take Wheeler's value for c, then the value of u from the last formula becomes equal to 5 ft. per second.

The agreement between the observed and theoretical values is remarkable.

Owing to the instability of motion in a bore the formulæ given above can only be regarded as of use in indicating conditions favourable to the formation and rapid propagation of bores. It has been found, as was indicated earlier in this chapter, that artificial changes in a river may suppress a bore. The reason for this is clearly revealed in the preceding investigations, for the object of training walls or embankments is to deepen the river, and since u^2 is normally less than gh it follows that if the depth of a river is increased then the chances of formation of a bore are lessened.

There are certain facts regarding bores which can be easily related to the theories. It has been shown that a bore is generated when certain restrictions or conformations of a channel bring about a critical stage of velocity of water. It has also been remarked that the velocities of water normally experienced are below the critical values. It is evident therefore that there is a greater possibility of a bore being caused when the depth of water is low, so that we should expect the greatest bores to occur after low water on a spring tide. Further, the greater the range of tide the greater is the probability of very small depths of water occurring at low water so that large bores are usually associated with large tides.

*27.9. Mathematical theory for the rate of propagation of a bore

If we consider the bore as having been formed, and reduce the motion to be considered to a steady state by the artifice of supposing ourselves to travel with the bore then we must have

$$(u_1 - c)^2 \lessdot gh \quad . \quad . \quad . \quad . \quad . \quad (27.9a)$$

$$(u_2 - c)^2 > gh \quad . \quad . \quad . \quad . \quad . \quad (27.9h)$$

* See par. 1, page vii.

where u_1 = velocity of current behind the bore
u_2 = velocity of current in front of the bore
c = rate of propagation of the bore
h = the mean depth.

We shall write

h_1 = the depth of water behind the bore
h_2 = the depth of water in front of the bore

so that

$$h = \tfrac{1}{2}(h_1 + h_2) \qquad \text{(27.9c)}$$

We shall take positions for two sectional planes so that we can avoid the unequal signs and write

$$(u_1 - c)^2 = (\sqrt{gh} - x)^2 \qquad \text{(27.9d)}$$

$$(u_2 - c)^2 = (\sqrt{gh} + x)^2 \qquad \text{(27.9e)}$$

where x is a convenient positive quantity enabling us to deal with formulæ (27.9a) and (27.9b) in a simple fashion.

Then the condition of equal rates of transport of volume requires

$$h_1(u_1 - c) = h_2(u_2 - c) \qquad \text{(27.9f)}$$

and on substitution in the previous formulæ we get

$$h_1(\sqrt{gh} - x) = h_2(\sqrt{gh} + x)$$

and therefore

$$x = \frac{h_1 - h_2}{h_1 + h_2}\sqrt{gh} \qquad \text{(27.9g)}$$

If we take $u_2 = 0$ then we have the case of a bore invading still water, and from (27.9e) and (27.9g) we can deduce

$$c = \sqrt{gh} + x = \frac{2h_1}{h_1 + h_2}\sqrt{gh} = \frac{h_1}{h}\sqrt{gh} \qquad \text{(27.9h)}$$

If, however, $u_2 = -\mathrm{U}$, in the case of a river current flowing against the bore, then

$$c = \frac{h_1}{h}\sqrt{gh} - \mathrm{U} \qquad \text{(27.9i)}$$

If we denote the height of the bore by B then we have

$$\mathrm{B} = h_1 - h_2 = 2(h_1 - h) \qquad \text{(27.9j)}$$

whence

$$h_1 = h + \tfrac{1}{2}\mathrm{B} \qquad \text{(27.9k)}$$

and therefore

$$c = \left(1 + \frac{1}{2}\frac{\mathrm{B}}{h}\right)\sqrt{gh} \qquad \text{(27.9l)}$$

CHAPTER XXVIII

SURGES

28.1. Surges of seismic origin

WE shall now proceed to discuss the phenomenon of the so-called tidal waves which are often experienced after large seismic disturbances. A classical case of such a wave is that which occurred when Lisbon was devastated by an earthquake in the year 1755. Any sudden local disturbance of the sea will cause a wave to spread in all directions, just as a stone cast into a pond will cause ripples to spread out in circles on the surface of the pond. Such waves are not tidal waves at all, for tides are recurring phenomena with well-defined regularities, whereas these seismic waves are casual and intermittent. The term *surge* is a better description of the phenomena.

The only point of interest in connection with the theory of tides is that such waves travel long distances and their rates of travel are easily obtained and demonstrate the law, which has become so familiar, that free progressive waves travel at a rate $c = \sqrt{gh}$.

28.2. Surges of meteorological origin

A class of surges of great interest and importance consists of those oscillations and perturbations of the sea level which arise from meteorological causes. It is unusual for the most carefully computed tidal predictions to be exact, owing to the fact that the sea surface is almost continuously disturbed by changes in air pressure and wind. The problem of forecasting these disturbances has engaged very much attention and it would be to the advantage of navigators if rules could be formulated which would enable forecasts to be made with reasonable accuracy, even 24 hours in advance. Whereas the surges of seismic origin are comparatively rare, the surges of meteorological origin are very frequent, but one lesson has been learnt from the former, which is, that surges generated in mid-ocean will travel rapidly (say, about 500 miles an hour) until their influences are made manifest on the coasts. Thus we should expect weather disturbances in mid-Atlantic to cause perturbations of the sea level and tides on the British coasts, as well as on all other coasts. It is to some extent a source of dismay to reflect that the meteorological effects of to-day may have had their origin yesterday in the southern Atlantic! But there is reassurance in this, that the effects fall off as the distance increases. The major surges experienced in British waters must therefore be generated much nearer home than mid-Atlantic.

We shall have to distinguish between the meteorological disturbances caused by steady or slowly-changing winds and air pressure and those caused by storms. A surge, in the proper sense of the word, is a water movement which is quickly generated and whose effects are soon over. The rapidity of generation, the notable rise (or fall) of the sea level, and the manner in which the surge travels, are characteristics which distinguish it from other meteorological disturbances of sea level.

Clearly the mechanism of generation and propagation is not unlike that of the bore, and we shall thus have to consider the subject in much the same way as in the previous chapter. But the variety of effects is so great that we can only indicate a few leading principles. A large amount of research work has yet to be done on this great problem. We shall consider in due order the effects of slow barometric changes, the influence of wind, and rapid changes in barometer and wind.

28.3. Effects of changes in barometric pressure

If we consider an enclosed sea, and ignore the tidal variations, then the level surface taken up by the sea with uniform atmospheric pressure will be independent

of the actual pressure. The surface will not suffer deformation if everywhere the pressure is uniformly increased. If, however, a permanent change is made in the distribution of pressure then the sea will respond to that change and the elevation will be increased under areas of low pressure and decreased under areas of high pressure. The sea, in fact, will act like an inverted water barometer, and in places where the mercury barometer is 1 inch higher than the mean over the sea, the water will be lowered 13 inches, since the density of water is 1/13th that of mercury.

Since the maximum excursion of the mercury barometer at any time from its mean value is only about 1·5 inches it follows that the maximum elevation or depression of water due to this cause will only be about 20 inches, but when we take account of the fact that in an enclosed sea the difference of pressure at any moment at any place is not at all likely to be as great as 1·5 inches from the mean over the sea, unless it is a very large sea, we conclude that the maximum disturbance in sea level due directly to changes in barometric pressure is not likely to exceed 12 to 15 inches.

Since water is almost incompressible it follows that changes in pressure are transmitted almost instantaneously, but it does not follow that the surface will instantaneously take up the equilibrium or static condition pertaining to the distribution of the pressure. In fact, we encounter precisely the same problem as that of the response of the oceans to the tide-generating forces. If these forces change infinitely slowly, or if the depth of the ocean is infinitely great, then the equilibrium relations will be established and maintained, hut as water has inertia the response to the quickly varying tide-generating forces depends upon the depth and configuration of the ocean. So also the response to barometric pressure depends upon the same conditions.

In the case of a small sea we are concerned with gradients of atmospheric pressure rather than with absolute values. In a large sea or ocean, however, we can assume that the mean pressure over the ocean at any time is almost invariable, and practically the same as the mean pressure at any one place over a period of time. Hence, this accounts for the approximate truth of the relation that the disturbance in sea level is approximately 13 times the departure of the barometric pressure from its mean value. In actuality, the factor is only appropriate to very slow changes in pressure, and where statistical investigations have included all manner of rates of variation, very great diversity in the factors has been found.

What we have said as to the response depending upon the inertia of the water in the same manner as to tidal forces accounts sufficiently for this, and we shall proceed to discuss the effects of the rapid changes in the barometer due to a travelling disturbance.

The mathematical theory of such changes will be discussed in the next article, but the general result of that theory is to indicate that if a pressure disturbance is travelling at a rate c over water of depth h, then

$$\left.\begin{matrix}\text{the elevation of}\\ \text{the surface}\end{matrix}\right\} = \frac{\text{the equilibrium relation}}{1 - c^2/gh} \quad . \quad . \quad . \quad (28.3\text{a})$$

In this formula, the "equilibrium relation" is the elevation that would be computed according to the principles outlined above, a change of 1 inch in the mercury barometer corresponding to a change of about 13 inches in the water. The formula shows that if c, the rate of travel of the pressure disturbance, approaches $\sqrt{gh}$, which is the rate of travel of a free wave in water of depth h, then the elevation of the sea will become much larger than the statical elevation.

The formula given above is of a type familiar to us by now, for a very similar formula was obtained in Chapter XXIII in connection with the response to tidal forces, and another one in connection with bores in Chapter XXVII. It is an illustration of a general principle that if the disturbing forces travel at the same rate as that of a free progressive wave in the fluid, then *resonance* will occur, but if the disturbing forces travel faster or slower than this special rate then the unlimited accumulation of energy does not take place.

In the case of the barometric disturbance the elevations of the sea surface will

correspond to the depressions of the barometer so long as c^2 is less than gh, but there will be a change as the critical value is exceeded, so that where we would naturally, according to the statical law, expect elevations, we get depressions. There is said to be an *inversion.*

We see now very clearly that slow changes in barometric pressure cannot be expected to yield such large results as quick changes, and that when the rate of travel of the atmospheric disturbance approaches the critical value $\sqrt{gh}$ a large surge will be generated. It is also obvious that on this theory there is no clearly marked distinction between surges and the static changes, but it is convenient to use the term surge when the phenomena are associated with rapidly moving pressure disturbances. The distinction is qualitative, rather than quantitative.

It should be noted that in connection with the resonance effects, the frictional forces which have been neglected will prevent unduly large responses to the forces. The simple formulæ can only indicate conditions likely to lead to large disturbances.

*28.4. Rapid changes in barometric pressure

In order to investigate the results of rapid changes in barometric pressure, we shall take a simple case of a barometric pressure disturbance travelling at a rate c over and along a channel of constant breadth and unlimited depth.

For the moment we shall consider the state of affairs in which both the pressure and the sea have settled down to a steady relationship; that is, we shall not at present discuss the generation and building up of the motion of the sea from an initial state of uniform pressure and undisturbed level.

Let the depth of the fluid be denoted by h, its density by ρ, the barometric pressure at the surface by B, and the internal pressure at any point in the fluid by p. Let c denote the rate at which the pressure disturbance is travelling along the channel, and let us suppose that we are travelling with the pressure disturbance so that the water appears to be moving in a steady state. Then Bernoulli's equation (17.9r) applied to a stream tube in the surface of the fluid gives

$$\frac{p}{\rho} + \tfrac{1}{2}(u - c)^2 + gy = \text{constant} \quad . \quad . \quad . \quad (28.4\text{a})$$

where y is the elevation of the surface, u is the actual velocity of the fluid, supposed to be constant from the surface to the bottom, and $(u - c)$ is the velocity we have to consider in the steady motion, since we are moving with the profile.

The condition of equal rates of transfer of volume across sectional planes gives

$$(u - c)\,(h + y) = \text{constant} \quad . \quad . \quad . \quad . \quad (28.4\text{b})$$

where h is the mean depth of the fluid.

If we ignore squares and products of the relatively small quantities u and y in these two equations we get

$$\frac{p}{\rho} + gy - uc = \text{constant} \quad . \quad . \quad . \quad . \quad (28.4\text{c})$$

since c^2 is a constant. Also at the surface the pressure p is equal to the barometric pressure B, so that we get

$$gy - uc = -\frac{\text{B}}{\rho} + \text{constant} \quad . \quad . \quad . \quad . \quad (28.4\text{d})$$

From (28.4b) we also get

$$uh - cy = \text{constant} = 0$$

since on the average u and y are zero so that the average value of $(uh - cy)$ must also be zero.

Hence we get, on elimination of u,

$$y\left(g - \frac{c^2}{h}\right) = -\frac{\text{B}}{\rho} + \text{constant}$$

* See par. 1, page vii.

We can write this as

$$y\left(g - \frac{c^2}{h}\right) = -\frac{B - B_0}{\rho} \quad . \quad . \quad . \quad . \quad . \quad (28.4e)$$

where B_0 is the mean barometric pressure.

If the disturbance is almost stationary, so that c is very small, we get the static condition

$$y = -\frac{B - B_0}{g\rho} \quad . \quad . \quad . \quad . \quad . \quad (28.4f)$$

This formula only expresses what we concluded in the previous article, that in the static condition the water surface will be depressed where the pressure is high, and the elevation will be strictly proportional to the local departure of pressure from the mean.

We can thus write (28.4e) in the form

$$\left.\begin{matrix}\text{elevation of}\\ \text{surface}\end{matrix}\right\} = \frac{\text{the equilibrium elevation}}{1 - c^2/gh} \quad . \quad . \quad (28.4g)$$

28.5. The development of pressure surges

In the preceding articles only the steady or equilibrium state was considered in which the régime of pressure and water movements was well established. The process of development of that steady state, from an initial state of a calm sea under uniform pressure, is somewhat complicated. In normal tidal theory the analogous problem of tracing the history of water movements, supposing the lunar forces to be brought suddenly into being, has never been attempted. In that theory we are dealing with forced movements and it has been supposed that all other movements have died away under the influence of frictional forces. It is clear, however, that if any forces are brought suddenly into being at a particular place they will immediately set up free oscillations which will travel outwards with a rate appropriate to the depth of the sea or ocean. In a large sea or ocean, and allowing for the gyration of the earth, the investigation of this forced motion and the free motion is exceedingly complex, but some idea of the resulting motion can be obtained by considering a narrow canal (so as to avoid gyrational complications), over and along which a pressure disturbance is suddenly created and thereafter travels at a uniform rate along the canal.

Suppose that we have a stationary pressure disturbance such that it produces a local elevation of the water surface as shown in Fig. 28.1 by the full line (A). An equal and opposite disturbance will produce (B). Thus calm and undisturbed conditions can be represented by these two effects annulling one another. Now let us suppose that the pressure disturbance producing (A) travels to the right with velocity c and that (A) is *maintained*. Its position after a certain interval of time is shown in Fig. 28.2. Also suppose that the pressure disturbance producing (B) suddenly ceases. The disturbance (B) will break up into two waves similar to (B) but with half the amplitude, and these will travel with the rate of a free wave $\sqrt{gh}$ in opposite directions, as in Fig. 28.2.

Fig. 28.2 has been drawn on the assumption that c is less than $\sqrt{gh}$. If we consider the effects of the travelling pressure disturbance, we see that the waves travelling in the same direction as the directions of travel of the pressure disturbance will be continually reinforced, while that which travels in the reverse direction will not be reinforced. Exact theory shows that the amplitude of the wave (A) will be proportional to $1/(1 - c^2/gh)$ and the amplitude of the waves (B_1) and (B_2) will be proportional to $1/(1 - c/\sqrt{gh})$ and $1/(1 + c/\sqrt{gh})$ respectively. The amplitudes of (A) and (B_1) will become very large when c^2 approaches the value gh, but the amplitude of (B_2) will never be large.

We are naturally most interested in the two waves (A) and (B_1) which travel to the right at rates c and $\sqrt{gh}$ respectively. At a point P well away from the initial

zone of disturbance we shall get different conditions according to whether c is less or greater than $\sqrt{gh}$. If, for instance, c is less than $\sqrt{gh}$ then the first effect of the sudden disturbance at the point C will be to produce a *depression* at P, followed later by the main *elevation*. The depression will be much smaller than the elevation. It

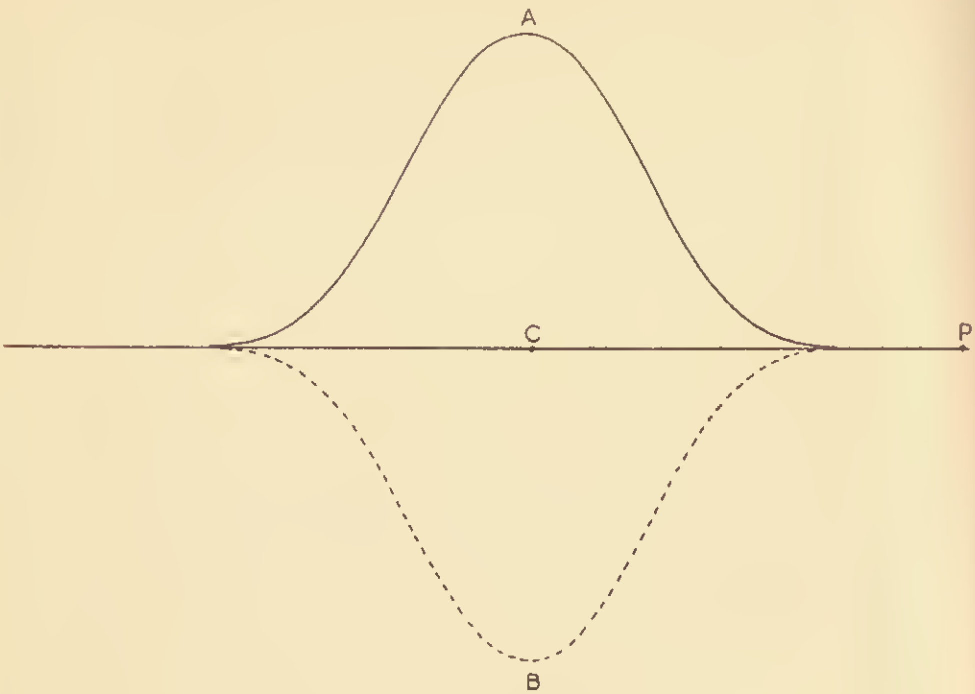

FIG. 28.1. Equal and opposite disturbances.

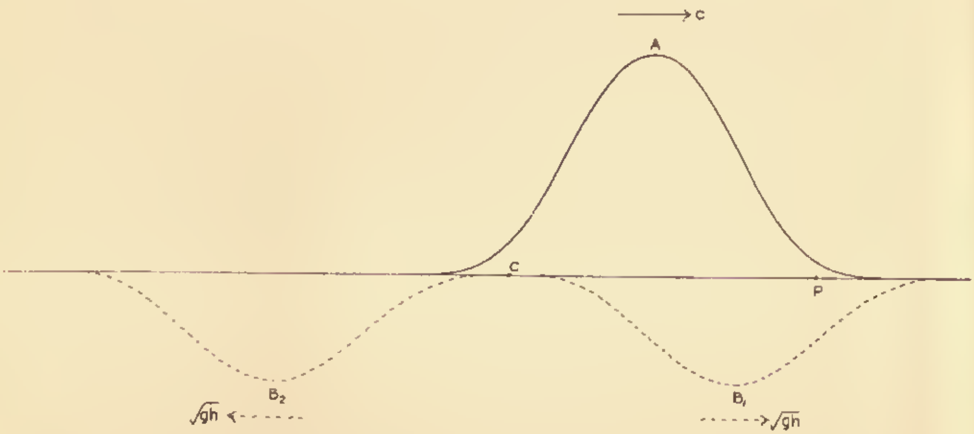

FIG. 28.2. One disturbance caused to travel and the other annulled.

has been noticed that the raising of sea level in an actual surge is often preceded by a lowering of level of smaller amount, and the above explanation helps to account for this.

If, however, the pressure disturbance travels at a faster rate than $\sqrt{gh}$, then the disturbance arrives first at a place and the effect on sea level is partly reversed when the secondary wave arrives.

The above investigation presupposes a sudden creation of pressure disturbance. In actuality, discontinuities of this kind are not very common, but they have been known to occur. If the pressure is changing rapidly then the effects on the sea will have some resemblance to those discussed above.

28.6. The influence of wind upon the sea

The investigation of the influence of wind on mean sea level is necessarily very complicated, and must take account of the physical properties of water in ways which we have not so far considered. A very brief exposition of these problems will be given, and then certain outstanding principles will be deduced.

The only influence that the wind can exert on the sea is clearly exerted through the frictional forces on the surface ; in other words, the friction between the air and the water retards the wind near the surface of the sea and also sets the surface water in motion, primarily in the same direction as the wind. By simple processes of traction the upper particles of water communicate motion to the particles at lower levels, so that the upper layers at first move faster than the lower layers. At the very bottom the friction on the sea bed must retard the lower layers of water.

The variation of velocity of water from top to bottom under the influence of wind depends upon the circumstances. If the sea is very large then the maximum velocity will be at the surface and the minimum velocity at the bottom, but if there is a coastal barrier other considerations need attention. If we suppose that the traction of the wind sets the water in motion towards the coast, then this process can only continue until the back pressure of the elevated water near the coast is sufficiently great to counterbalance the tractive forces. A *circulation* of water may then be maintained, the upper layers moving continuously forward and the lower layers continuously backward.

It is evident that the effect of the wind is to impart energy to the water in the sea. It was pointed out in connection with the study of progressive waves and standing oscillations that the water in a channel absorbed energy according to its own laws, and that the result was largely independent of the manner in which the energy was supplied. The same principle applies to this new problem, and the energy abstracted by the sea from the wind is absorbed in the quickest possible manner. Part of the energy is used up in imparting motion as described above, but another part is used in setting up waves upon the surface of the sea. The existence of this wave motion naturally facilitates the transference of energy as outlined above, and a limiting stage will be reached when the supply of energy from the wind equals that dissipated by friction. Until that stage is reached the waves will increase in size. The whole tendency of friction is to absorb energy or to resist changes in the existing régime, and the corrugation of the sea surface by the wave motion, in effect, increases the frictional force of the wind on the sea surface, and quickens the response of the sea. It is evident that the presence of the waves induces a new effect, that of actual forward *pressure*, as distinct from surface traction, and this pressure is on the rear surface of the wave.

The waves so set in motion will travel at a rate which depends only upon the depth if the wave is a long wave, but upon the depth and also the wave-length if it is a short wave. (A long wave is defined as one in which the wave-length is much greater than the depth of water.) But if the waves produced travel at a greater rate than the wind then actually they pass out of its influence and decay. For the waves to increase they must travel more slowly than the wind. It is clear, however, that there may be generated a system of waves which will travel ahead of the main effects of the wind, and such waves may act as the forerunners of a wind surge. The advent of cyclonic disturbances is indeed often heralded by a long swell.

Further detailed investigation of the mechanism of the wind effects is outside the scope of this Manual. The subject is a vast one and requires much more investigation than it has yet received. We shall confine ourselves therefore to some cases of

wind effects which can be considered by relatively simple mathematics, as in the next article, where it is shown that if a localised atmospheric disturbance travels with a velocity c, and the winds associated with the disturbance communicate energy to the sea, then the slope of the sea surface in the direction of the wind is given by

$$\frac{\text{the equilibrium gradient}}{1 - c^2/gh} \quad . \quad . \quad . \quad . \quad (28.6a)$$

Here the equilibrium gradient is that which would be generated if $c = 0$.

This equation is very much like that given in (28.3a) for the increase in elevation due to a travelling pressure disturbance.

We see at once that whatever be the increase in energy, or whatever be the form it takes, so long as it is positive (as it must be), then the elevation increases in front of the disturbance so long as c is less than $\sqrt{gh}$. When the velocity approaches the critical value we tend to get infinitely large values—in actuality, if friction is adequately allowed for, there could not be infinitely large elevations, but the formula shows what the tendency is.

Thus just as a pressure disturbance travelling at a rate approaching $\sqrt{gh}$ would yield a pressure-surge, so also a wind-system travelling at a rate approaching $\sqrt{gh}$ will yield a wind-surge.

A *sudden* rise of wind could be investigated as in Art. 28.5, but there is a difference between the cases of wind and pressure. Discontinuities of pressure, or sudden changes in pressure, yield immediate effects owing to the incompressibility of water, but a wind requires time to act from the surface downwards. Thus in general we should expect the effects of changes of pressure to be experienced without much delay and the effects of changes in wind to be felt after an appreciable time-lag.

*28.7. Complex surges

It was shown in the previous article that the principal effect of the wind is to communicate energy to the water beneath it. We shall now ignore the mechanism of this, as involving the waves set up, and the variation of velocity from the surface to the bottom, and we shall consider the motions in an infinitely long channel (with a level bed) along which a complex atmospheric disturbance is travelling. We shall not consider the structure of the disturbance, but we shall suppose it to be localised. Our object is to generalise the investigation of Art. 28.5 so as to include pressure effects and wind effects.

We shall adopt the usual artifice of considering relative motions, so that we shall imagine ourselves to be travelling with the disturbance. The water will then appear to be flowing backwards with a mean velocity $c - u$, where u is the true mean velocity from the surface to the bottom. This artifice allows us to deal with progressive effects.

Let now sectional planes be taken across the channel on either side of the plane in which we are supposed to be travelling. Let the elevations of the surface above the level bed be $h - h'$, $h + h'$, where h is the elevation of the surface in the mean position ; also let the apparent velocities of fluid be $u - u' - c$, $u + u' - c$ in the forward direction. Then the condition of equal rates of transfer of water requires :

$$(h - h')\,(u - u' - c) = h(u - c) = (h + h')\,(u + u' - c) \quad . \quad (28.7a)$$

If we ignore squares and products of the relatively small quantities h' and u' we obtain

$$h'(u - c) + u'h = 0 \quad . \quad . \quad . \quad . \quad (28.7b)$$

If we remember that Bernoulli's equation (17.9r) is concerned with energy relations, then we can denote by E the increase in energy between the two planes as derived from the atmospheric disturbance, so that we get

$$\tfrac{1}{2}\,(u - u' - c)^2 + g(h - h') = \tfrac{1}{2}\,(u + u' - c)^2 + g(h + h') - \mathrm{E} \quad . \quad (28.7c)$$

* See par. 1, page vii.

Again ignore squares and products of the relatively small quantities h' and u' and we obtain

$$u'(u - c) + gh' = \tfrac{1}{2}\mathrm{E} \quad . \quad . \quad . \quad . \quad (28.7\mathrm{d})$$

Substituting for u' from (28.7b) then gives

$$h'\left\{g - \frac{(u - c)^2}{h}\right\} = \tfrac{1}{2}\mathrm{E} \quad . \quad . \quad . \quad . \quad (28.7\mathrm{e})$$

Now the velocity of the water will generally be very small relatively to the velocity of the disturbance or to $\sqrt{gh}$, so that we can now ignore u in the above equation and obtain the simple expression

$$h' = \frac{\tfrac{1}{2}\mathrm{E}h}{gh - c^2} \quad . \quad . \quad . \quad . \quad (28.7\mathrm{f})$$

If the value of h is very great, then

$$\tfrac{1}{2}\mathrm{E}/g$$

represents the change in level between the two places in question in the case of a very deep channel, so that we can express the above formula in the alternative form

$$\left.\begin{array}{c}\text{gradient of sea}\\ \text{surface}\end{array}\right\} = \frac{\text{the equilibrium gradient}}{1 - c^2/gh} \quad . \quad . \quad (28.7\mathrm{g})$$

as in the analogous case of the pressure disturbance.

28.8. Wind currents in deep water

The investigation of the effects of wind on the water in a very large sea is much more difficult than that previously considered for a case of an infinitely long channel, but we shall briefly indicate the principal results.

If there are no external forces operating other than those due to the wind then the water will be transported in the same direction as the wind. Owing to the internal frictional forces the velocity of water will decrease from the surface downwards, and after a certain depth depending on the amount of friction the motion will be insensible. So long as the water can be transported in the direction of the wind this will characterise the motion, but if the wind-current is barred by a coast then the water will be accumulated at the coast and cause a back-pressure which will ultimately cause a steady state to be reached, in which water is conveyed away as quickly as it is brought to the coast; *e.g.*, a circulation may be set up, with a negative current below the positive wind-driven current at the surface.

In order to avoid complications of this sort, we shall consider the sea as being so large that the coastal effects can be left out of consideration while we investigate the effects of other forces such as those due to the gyration of the earth and internal friction.

We saw in Art. 20.2 that if an element of fluid has a velocity in space then there is exerted upon it a force due to gyration, and this force is at right angles to the direction of velocity at the moment. Hence the gyroscopical forces will tend to cause, on the whole, a drift of water to the right of the wind in the northern hemisphere.

The frictional forces will cause a progressive diminution of the velocity of current as we go from the surface to the bottom.

The resultant distribution of velocity is pictured diagrammatically in Fig. 28.3.

According to the mathematical theory given by Ekman, the drift at the surface in a steady state in the northern hemisphere is inclined at an angle of 45° to the right of the direction of the wind, and the resultant drift (the average of all values from surface to bottom) is at right angles to the wind. It must be remembered, however, that in obtaining this result the effects of coasts have been entirely ignored. In fact, when allowance is made for coastal effects, the angles of surface

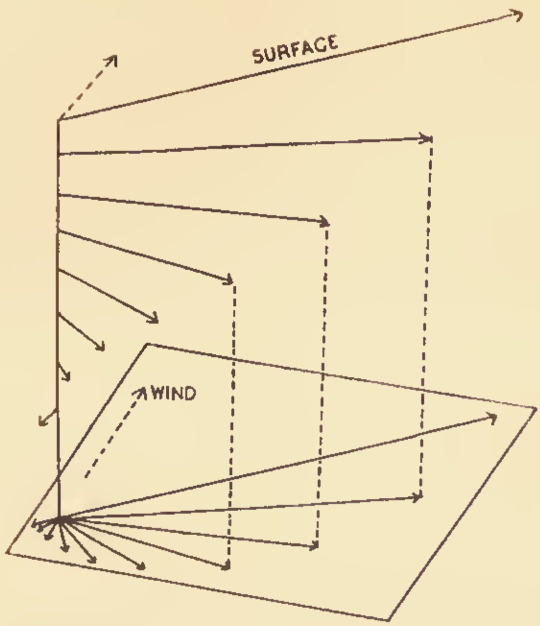

FIG. 28.3. Currents under influence of wind ; direction and velocity at different depths in the northern hemisphere according to Ekman.

drift and resultant drift are considerably smaller, a frequent value for the angle of surface drift relative to the direction of the wind being about 20°.

In the southern hemisphere the drift will be to the left of the wind.

28.9. Surges due to cyclonic systems

In a cyclonic system of distribution of barometer and wind in the northern hemisphere, a low pressure at the centre of the cyclonic system is accompanied by

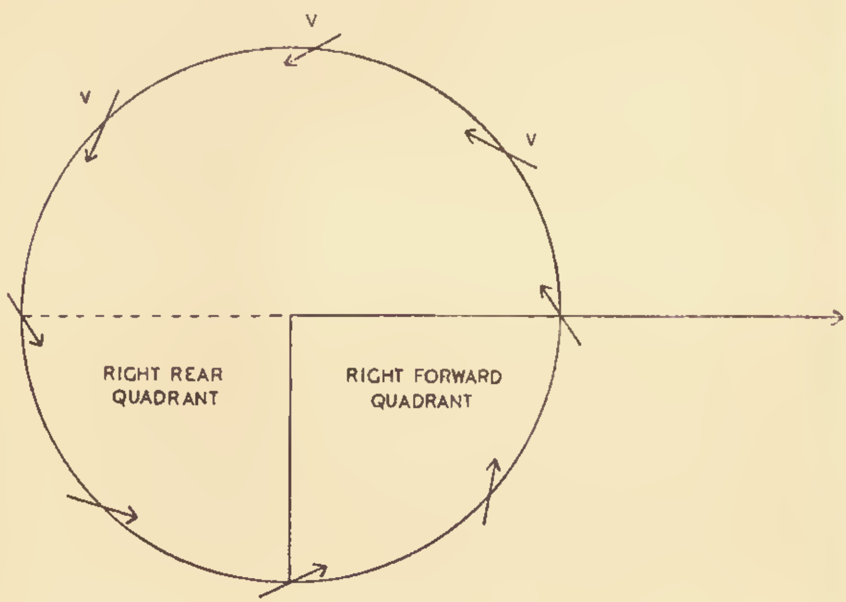

FIG. 28.4. Direction of winds in travelling cyclone.

winds blowing in an anti-clockwise direction round the centre. The winds do not blow at right angles to radii from the centre of pressure but are directed slightly inwards, as shown in Fig. 28.4. We shall suppose that the whole system is travelling eastward at a rate c, and that the velocity of wind in an isobaric circle (*i.e.*, at equal distances from the centre of pressure) is denoted by V.

Then as the pressure system moves along, provided c is less than $\sqrt{gh}$, we have the sea disturbance moving with the pressure disturbance, so that the sea is high in the centre of the cyclone, it is rising on the forward edge and falling on the rear edge. These conditions are given in Fig. 28.5.

Now consider the effects of the wind. On the right side of the cyclone the wind is causing the sea to rise more in the right forward quadrant than in the right

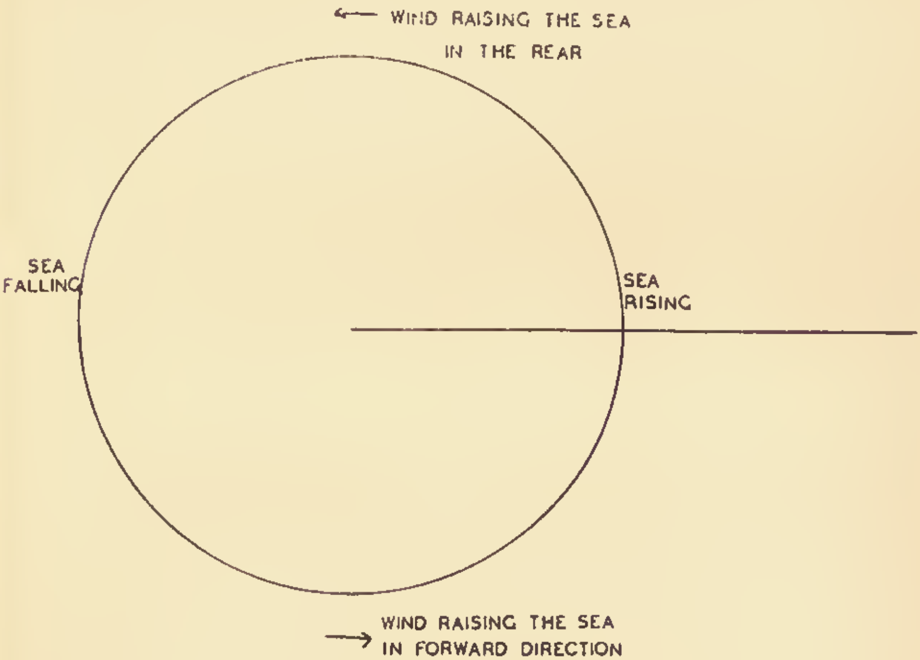

Fig. 28.5. Effects of winds in travelling cyclone.

rear quadrant. Thus the wind is helping the pressure disturbance in the right forward quadrant.

On the left side of the system the wind is opposing the pressure effects. Hence the greatest storm-surges must be expected to be found on the right of a travelling cyclone. Further, as this surge tends to travel and to produce forerunners travelling at the rate $\sqrt{gh}$ the effects of the earth's gyration will be to accentuate movement to the right of the path of the cyclone.

Again, owing to the winds being directed slightly inward, the components of wind in the direction of travel of the cyclone are greater in the right rear quadrant, so that on the whole we should expect these winds to be more effective than those in the forward quadrant.

In the cyclonic system in the southern hemisphere the winds blow in a clockwise direction round the centre; the description above may, however, be utilised by interchanging the words "left" and "right."

28.10. Storm surges in the North sea

The occurrence of disastrous flooding in the Thames in January, 1928, owing to a storm surge in the North sea caused investigations to be made as to the character

and causes of such surges. The actual surge was traced from Dunbar in Scotland to Southend on the Thames estuary and thence to the coasts of Belgium, Holland and Denmark. The rate of travel of the surge appeared to be that of a free wave. Many other surges were traced and in most cases the same result was obtained, that the earliest indication of a surge is found on the east coast of Scotland and that this surge appears to travel round the North sea in an anti-clockwise direction, increasing in magnitude until the Flemish bight is reached and thereafter decreasing. Some surges were traced in this manner right into the Baltic sea.

The conclusion was drawn that the surges may have originated in atmospheric disturbances of the sea to the north or north-east of Scotland, and that they travelled round the North sea as free progressive waves in an anti-clockwise direction. The normal tidal movement is also of this character *on the coasts*, but it was shown in Chapter XXV that this apparent movement is true only for the coastal phenomena and that the actual movements are more complex than could be explained by the theory of simple progressive waves. Similarly the propagation of a surge probably involves more complicated explanations than that first put forward.

Recent work by Corkan has shown that the surges at Southend can actually be computed with great accuracy by using a formula due to Proudman and Doodson. This formula was obtained mathematically for the elevation of a water surface in a

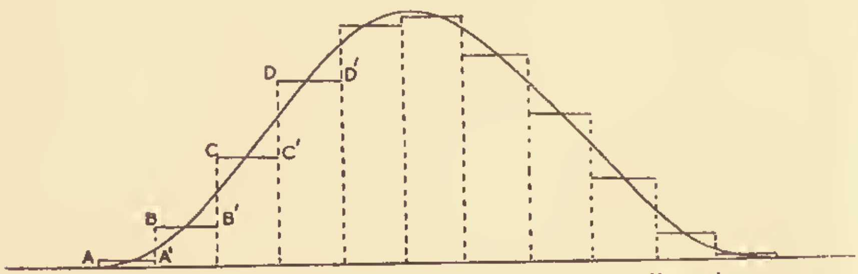

FIG. 28.6. Representation of continuous variation by short discontinuous ones.

channel of constant depth, closed at both ends, but the following explanation will serve to show the underlying principles.

Suppose that a wind of velocity V begins suddenly to blow along the channel. There will be a drift of water under its influence and ultimately a state of equilibrium will be reached in which the water will be elevated at one end and depressed at the other under the influence of the wind. But as soon as the wind begins to blow the impulses imparted to the water set it in motion so that it oscillates in its own natural period. Hence the approach to the equilibrium state takes place at the same time as the free oscillation. If, however, we take into account the frictional forces which will oppose the oscillating motions we see that the amplitude of the oscillation will steadily decrease with the lapse of time. We have a *free damped oscillation* superposed on the equilibrium elevation so that the elevation of the surface above mean sea level at the head of the channel can be expressed by :—

$$y = \bar{y} + \mathrm{C} \cos (nt - k)\, e^{-dt} \qquad . \quad . \quad . \ (28.10a)$$

where e is the base of natural logarithms, n depends only upon the dimensions of the basin, t is the time measured from the time when the wind begins to blow, d depends upon the degree of friction and C, k are constants partly depending upon the velocity of the wind and the dimensions of the basin.

Formula (28.10a) is of a general type applicable to many physical problems in which free oscillations can occur in a frictional medium.

Suppose, however, that after a short interval of time another wind is suddenly caused to blow with velocity V, in the opposite direction, then its effects at the same point at the head of the channel will be given by :—

$$y = -\bar{y} - \mathrm{C} \cos \{n(t - t') - k\} e^{-d(t-t')} \ . \quad . \quad . \ (28.10b)$$

If, therefore, we consider both winds blowing together the second counteracts the first after the time interval t' so that there is a calm and the net result is that of a wind of velocity V suddenly caused to blow at time $t = 0$ and suddenly ceasing to blow at time $t = t'$. By adding together the formulæ (28.10a) and (28.10b) we can express the elevation of the sea.

Clearly we can take a series of such cases as in Fig. 28.6 where the actual wind velocity is indicated by the full line and it is considered for practical purposes as being represented by the broken line AA′BB′CC′ . . . The formula just referred to can now be applied to give the effect of the wind of mean velocity indicated by AA′, then to the effect of a wind indicated by BB′ blowing from time $t = t'$ to $2t'$, and so on, and the sum gives the effect on the sea level at any subsequent time.

In applying this formula the constants C, n, k, d can be obtained by simple inspection or trial from the observations. Thus from the tide gauge the differences between observed and predicted tide can be extracted to represent the surge. The

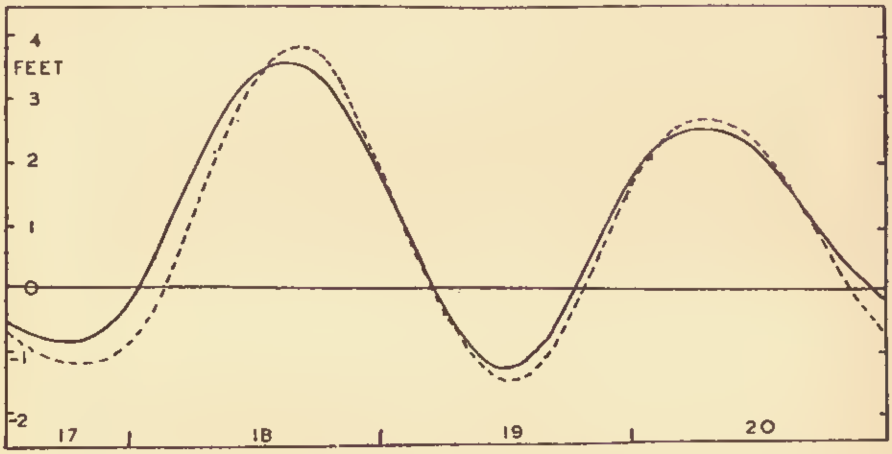

FIG. 28.7. Example of application of formula for computing storm-surges.
———— Recorded surge; - - - - - - Computed surge.

curve shows an oscillation which dies away, and a little trial with the factor soon gives the value of d. The period of the oscillation gives n, while C and k can be obtained by trial.

In this way, but using the appropriate mathematical methods, Corkan has applied the formula to the surges at Southend with remarkable and consistent results. Fig. 28.7 shows an actual surge at Southend together with the computed value. In applying the formula he used the winds over the North sea, and made the proper allowances for the variable depth and the direction of transport of water relative to the direction of the wind. He found the best free period for the sea to be about 36 hours and the ratio of successive amplitudes to be 1·75. He has examined many cases where the atmospheric disturbance covers an area of the same order as that of the North sea, and finds the formula very dependable.

This investigation at the time of writing is still in progress, but it is undoubtedly very promising and when it is extended to other places on the coast of the sea we can hope to get a greater insight into the phenomenon.

*28.11. Ekman's theory of drift-currents

The main results of this theory have been given in Art. 28.8 but it is desirable to give a more detailed exposition of the underlying principles. The original theory,

* See par. 1, page vii.

of course, is too highly mathematical for this Manual, but the following explanation, based on some notes by Proudman, brings out the fundamental physical principles.

The forces we have to consider when the water is very deep are practically limited to (1) the tractive force of the wind on the surface, (2) the gyroscopic forces tending to deflect the currents to the right of their path (in the northern hemisphere, which we shall take for discussion, and (3) the internal forces imparting motion from the surface to the lower depths of water.

Under the influence of these forces we can conceive of a large number of strata, in three of which the velocities of fluid may be denoted by V_1 for the uppermost stratum, V_2 for the middle stratum and V_3 for the lower stratum of the triad. We conceive these strata to be sliding as entities relative to one another. It is reasonable to assume that the velocity V_3 is related to the velocity V_2 in the same way as the velocity V_2 is related to V_1. This necessarily assumes that the forces are all proportional to the velocities, which may not be true of the frictional internal forces,

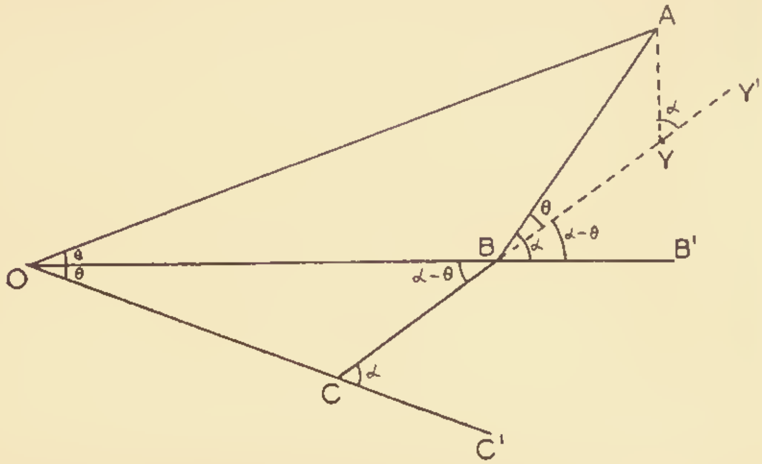

FIG. 28.8. Direction and magnitude of frictional forces in adjacent strata of water.

but the assumption is a reasonable one for the purpose of eliciting the character of the motions. From these assumptions we therefore deduce :

$$\frac{V_3}{V_2} = \frac{V_2}{V_1} = r, \text{ say}$$

and the directions of V_3 and V_2 differ from one another by the same angle (θ, say) as the directions of V_2 and V_1.

In Fig. 28.8 let OA, OB, OC represent in direction and magnitude the three velocities V_1, V_2, V_3, and let OB, OC be produced to B′ and C′. Then by the assumption made,

$$\text{BOC} = \text{AOB} = \theta \quad . \quad . \quad . \quad . \quad (28.11\text{a})$$

$$\text{OC} : \text{OB} = \text{OB} : \text{OA} = r \quad . \quad . \quad . \quad . \quad (28.11\text{b})$$

so that the two triangles are similar triangles, whence it follows that

$$\text{CB} : \text{BA} = r \quad . \quad . \quad . \quad . \quad (28.11\text{c})$$

and

$$\text{C}'\text{CB} = \text{B}'\text{BA} = \alpha, \text{ say} \quad . \quad . \quad . \quad . \quad (28.11\text{d})$$

Now the triangle AOB is a triangle representing velocities, so that BA represents in direction and magnitude the change in velocity in passing from the middle stratum to the uppermost one, and similarly CB represents in direction and magnitude the change in velocity in passing from the lowest stratum to the centre stratum. But changes in velocity are directly proportional to the forces giving rise to them and

therefore CB can be taken, on an appropriate scale, to represent in magnitude and direction the force exerted on the lowest stratum by the middle one, while BA represents in direction and magnitude the force exerted by the upper stratum on the middle one.

Since action and reaction are equal and opposite then the force exerted by the lower stratum upon the middle one is equal and opposite to that exerted by the middle stratum upon the lower one, which we showed above to be represented by CB. Let CB, therefore, be produced to Y where BY is made equal to CB. Then YB represents the force exerted by the lower stratum upon the middle one, while BA represents the force exerted by the upper stratum upon the middle one. Therefore, by the triangle of forces, YA represents the resultant force exerted upon the middle stratum by the two adjacent strata.

By reversal of thought, the force exerted by the middle stratum upon the adjacent strata is given in direction and magnitude by AY, and the question arises as to what force can be directly due to this middle stratum. Now, if there were no gyroscopic forces, the three velocities would all have the same direction, and there could be no resultant force exerted by the middle stratum upon the adjacent strata except a small frictional force. The principal force inherent to the middle stratum is the gyroscopic force arising from the velocity of the fluid in the stratum, and this we know to be at right angles to the direction of the stratum. It follows that the force AY must be at right angles to the direction of velocity OB, and directed to the right of the direction of OB.

We shall proceed to examine the direction of AY relatively to OB. Let BY be produced to Y′. Now since the angle B′BY is equal to the angle CBO which is equal to $(\alpha - \theta)$, and since angle B′BA is equal to α, it follows that angle YBA is equal to θ. But we already know from (28.11c) and (28.11b) that

$$BY : BA = CB : BA = OB : OA \qquad (28.11e)$$

so that it follows that the triangle YBA is similar to the triangle BOA, and therefore angle Y′YA is equal to angle B′BA ; that is,

$$\text{angle } Y'YA = \alpha \qquad (28.11f)$$

The inclination of AY to OB is equal to the sum of the two angles B′BY and Y′YA and according to the physical reasons given above it must be a right angle. Therefore

$$2\alpha - \theta = 90^\circ \qquad (28.11g)$$

Now the angle θ is clearly dependent upon the thickness of the strata, and with infinitely thin strata the limiting value of this equation gives :

$$\alpha = 45^\circ \qquad (28.11h)$$

But this angle is the angle between the velocity of the stratum and the tractive force exercised upon it by the stratum above, and since the tractive force at the surface is due to the wind and is in the same direction as the wind, it follows that the surface flow is inclined at an angle of 45° to the direction of the wind.

If we redraw Fig. 28.8 for small values of θ and join the terminal points A, B, C, D, E . . . of all radii, we get a curve like that of Fig. 28.9, which gives the curve of velocities against the direction of flow relative to the direction of the wind. Let this angle be denoted by ϕ. This curve is such that its tangent makes an angle of 45° with the radius, and such a curve is called an " equiangular spiral," whose equation can be shown to be :

$$V = V_0 e^{-\left(\phi - \frac{\pi}{4}\right)} \qquad (28.11i)$$

where V is the velocity, ϕ is measured in radians clockwise from the direction of the wind, V_0 is the surface velocity and e is the base of natural logarithms. The values of V/V_0 for values of ϕ at intervals of 10° (0.17453 radian) diminish in the ratio

$$r = e^{-0{\cdot}17453} = 0{\cdot}8399 \qquad (28.11j)$$

From this ratio we get the successive values of V/V_0 in the following table, which also gives values of $(V/V_0)\cos\phi$ and $(V/V_0)\sin\phi$, the components of velocity along the direction of the wind and at right angles to it.

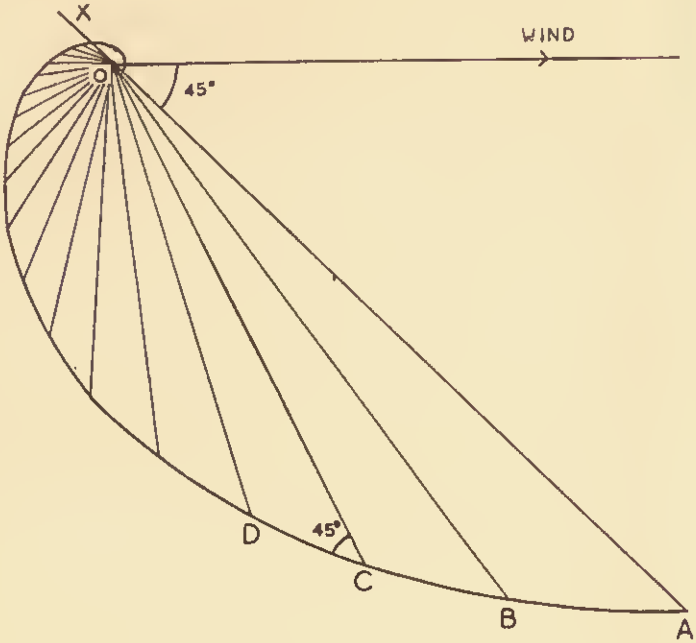

FIG. 28.9. Computation of directions and velocities for Ekman's diagram.

ϕ	$\frac{V}{V_0}$	$\frac{V}{V_0}\cos\phi$	$\frac{V}{V_0}\sin\phi$	ϕ	$\frac{V}{V_0}$	$\frac{V}{V_0}\cos\phi$	$\frac{V}{V_0}\sin\phi$
45°	1·000	0·71	0·71	145°	0·175	— 0·14	0·10
55°	0·840	0·48	0·69	155°	0·147	— 0·13	0·06
65°	0·705	0·30	0·64	165°	0·123	— 0·12	0·03
75°	0·592	0·15	0·57	175°	0·103	— 0·10	0·01
85°	0·498	0·04	0·50	185°	0·087	— 0·09	— 0·01
95°	0·418	— 0·04	0·41	195°	0·073	— 0·07	— 0·02
105°	0·351	— 0·09	0·34	205°	0·061	— 0·06	— 0·03
115°	0·295	— 0·12	0·27	215°	0·051	— 0·04	— 0·03
125°	0·248	— 0·14	0·20	225°	0·043	— 0·03	— 0·03
135°	0·208	— 0·15	0·15				

It will be noted that the table covers half a turn from A to X, and that the table of V/V_0 can be repeated for the next half turn by multiplying by 0·043 throughout.

The most important deduction from the table is that the mean value of

$$(V/V_0)\cos\phi$$

tends to zero, and if the table is continued far enough, and with more frequent values, it can be readily verified that the mean value is zero. This means that in very deep water the average component of current along the direction of the wind is zero; that is, the total transport of water is in a direction at right angles to the wind.

CHAPTER XXIX

SEICHES AND INTERNAL WAVES

29.1. Seiches

A SECONDARY effect of atmospheric disturbances is that free oscillations are set up in the oceans, in gulfs and in bays. The theory of standing oscillations given in Chapter XVIII shows that in an enclosed basin free oscillations are possible with periods depending upon tbe dimensions of the basin. Such oscillations occur in lakes, and these have been studied somewhat intensively, and the term *seiches* is applied to them. It is not necessary for us to amplify the theory for basins of all conceivable shapes, geometrical or as in nature, for the simple theory of free oscillations or seiches, as developed in Arts. 18.1 and 18.2 is sufficient. In fact, if the actual contour of a lake is expressed as nearly as possible by a rectangle then the formula (18.2i) will give approximate values for the periods of oscillations. The principal period increases with the length of the lake. If an observing station is situated half-way along a lake it will be on a nodal line for the principal longitudinal oscillation, but longitudinal oscillations of half this wave period (see Art. 18.4) and oscillations across the breadth of the lake will also be experienced.

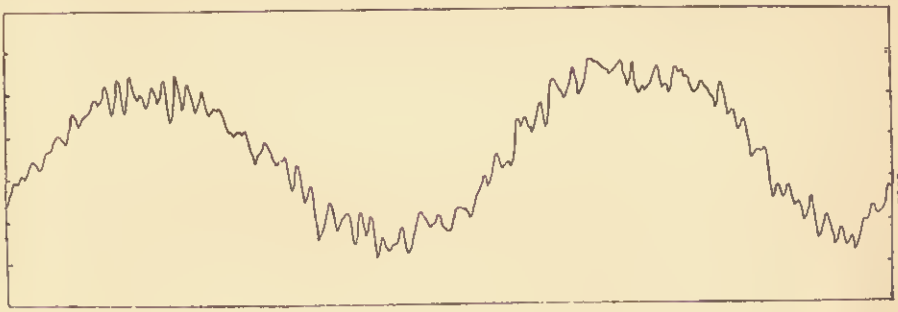

FIG. 29.1. Example of seiche motion superposed on tidal oscillation.

Witb regard to a gulf or bay, free oscillations are also likely to occur, according to the length and breadth of the gulf. Those which occur along the gulf will quickly lose energy by radiation into the ocean, but those across the gulf may continue to be reflected backwards and forwards for a comparatively long time until the energy of the motion is dissipated by frictional forces.

These seiches are set in motion, of course, by any disturbance of the water, so that atmospheric disturbances often result in these local seiches. Fig. 29.1 shows a seiche motion recorded by a tide gauge. The quick oscillations are due to seiches in a bay of small dimensions.

29.2. Seiches due to a submarine shelf

The theory of Art. 19.2 applied to a gulf of constant depth indicates that in any semi-enclosed area there will be local oscillations. Similarly, if there is an area where the sea-bottom suddenly rises to give a coastal shelf of relatively small depth, then oscillations may be set up in the region. As a general rule these oscillations must be expected to have longer periods than those experienced in bays, as the shelf will be larger than the average bay. It is unnecessary to consider the theory in any detail in this manner, but if seiches occur without obvious reason in a tide gauge diagram the possibilities of their being caused by a submarine shelf should not be ignored.

29.3. The phenomena of internal waves

The effects of a solid shelf in setting up seiches lead us to the consideration of the possible effects of strata of fluid of different densities. Some very remarkable phenomena have been observed in cases where such strata occur, and as these have some bearing upon the practical problems of tidal motion it is desirable to discuss them in some detail.

Strata of different densities may occur from two principal causes, the first being associated with salinity, the second with temperature. A layer of fresh water on salt water is not uncommon, and water flowing from different regions may have very different temperatures. It is not surprising that this subject began to receive prominence in connection with the discussions of observations made in polar and sub-polar regions.

The phenomena appear first to have been brought into prominence as a result of the Norwegian North Polar expedition of 1894. Observations of the salinity of water at different depths in deep water revealed a change of density at a depth of about 100 to 150 fathoms, and that the surface of discontinuity of density had a vertical rise and fall through a range of 10 to 15 fathoms. Such an internal wave must necessarily involve the transport of large quantities of water and therefore streams of high velocity may be experienced, with a very marked change in the velocity and direction of stream from one stratum to another. Such streams, of course, may balance one another, so that the total transport of water may only be associated with small variations of elevation of the surface.

The principal consequence is that very great caution is necessary in observing tidal streams, and in using the observations for the construction of charts of cotidal lines (as in Chapter XXIV), if there is any stratification.

It is only in deep water where the tides are small that the phenomena are of importance. In shallow seas such as those around the British Isles the tidal motion is so great and the depth of the sea so small that any tendency to stratification must quickly be overcome by the turbulent character of the tidal motion.

*29.4. The theory of internal waves

We shall content ourselves with considering a canal of uniform breadth, constant depth, and unlimited length, in which there are two strata of fluid, the upper having a mean depth of h_1 and the lower one a mean depth of h_2, so that the mean depth of the fluid as a whole is $(h_1 + h_2)$. We shall also suppose that a wave is travelling along the surface of separation of the two fluids, with a velocity c. Let the densities be denoted by ρ_1 and ρ_2 respectively.

Let us suppose ourselves to be travelling with the wave profile with the velocity c, and as in Fig. 29.2 let the elevation of the free surface above its mean value be denoted by y_1. Also let the elevation of the surface of separation of the fluids above its mean value be denoted by y_2.

Then Bernoulli's equation (17.9r), applied to the free surface where the atmospheric pressure is supposed to be constant, gives

$$gy_1 + \tfrac{1}{2}(u_1 - c)^2 = \text{constant} \qquad (29.4a)$$

where u_1 is the actual velocity of the fluid in the upper stratum, supposed constant from top to bottom, and $(u_1 - c)$ is the velocity relative to the observing point which is moving with the profile. At the surface of separation, if p is the pressure, then

$$\frac{p}{\rho_1} + gy_1 + \tfrac{1}{2}(u_1 - c)^2 = \text{constant} \qquad (29.4b)$$

and as the pressure is equal to the constant pressure of the atmosphere plus that equal to a weight of water through a depth between the two surfaces, we have

$$p = g\rho_1(y_1 + h_1 - y_2) \qquad (29.4c)$$

Hence we get

$$g(y_1 + h_1) + \tfrac{1}{2}(u_1 - c)^2 = \text{constant}$$

This is the same equation as in (29.4a) since gh_1 is a constant.

* See par. 1, page vii.

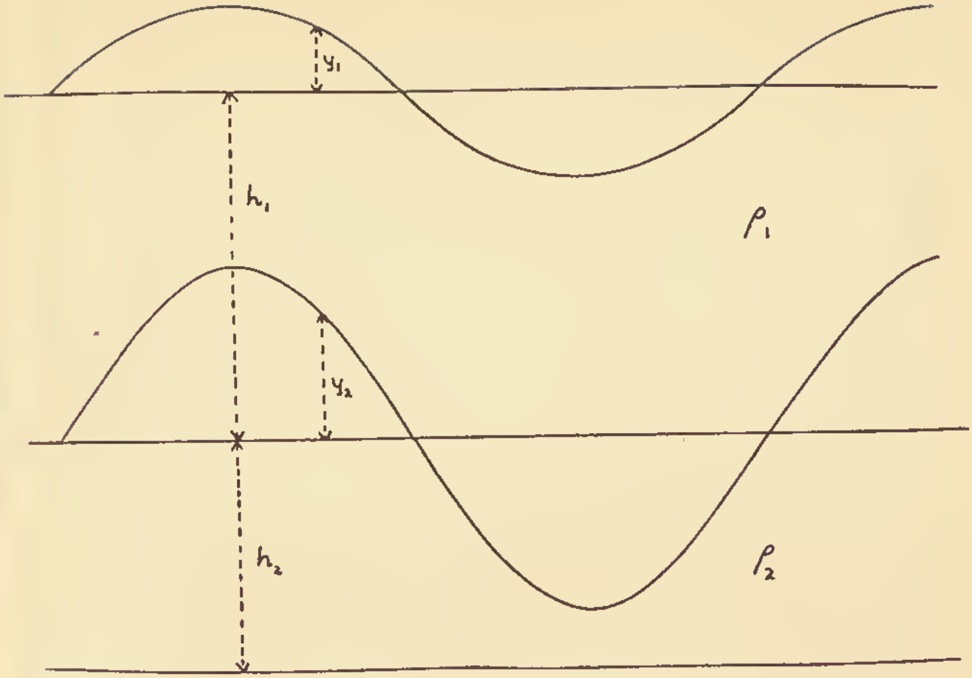

FIG. 29.2. **Elevations and densities in adjacent strata.**

Now consider the lower stratum and apply Bernoulli's equation at the surface of separation. We then get

$$\frac{p}{\rho_2} + gy_2 + \tfrac{1}{2}(u_2 - c)^2 = \text{constant} \quad . \quad . \quad . \quad (29.4\text{d})$$

and the pressure p has already been obtained in (29.4c), so that

$$\frac{g\rho_1(y_1 + h_1 - y_2)}{\rho_2} + gy_2 + \tfrac{1}{2}(u_2 - c)^2 = \text{constant}$$

If we absorb gh_1 in the constant, then

$$g\frac{\rho_1}{\rho_2}y_1 + gy_2\left(1 - \frac{\rho_1}{\rho_2}\right) + \tfrac{1}{2}(u_2 - c)^2 = \text{constant} \quad . \quad . \quad (29.4\text{e})$$

If in (29.4a) we absorb $\frac{1}{2}c^2$ into the constant, and ignore $\frac{1}{2}u^2{}_1$ since u_1 is supposed to be small compared with c, we deduce

$$gy_1 - u_1c = \text{constant} = 0 \quad . \quad . \quad . \quad . \quad (29.4\text{f})$$

where the constant is zero since the mean value of all such expressions on the left must be zero, on account of y_1 and u_1 fluctuating about mean zero values. Similarly, from (29.4e)

$$g\frac{\rho_1}{\rho_2}y_1 + gy_2\left(1 - \frac{\rho_1}{\rho_2}\right) - u_2c = 0 \quad . \quad . \quad . \quad (29.4\text{g})$$

Now consider the rates of transfer of volume across sectional planes, and we find

$$(u_1 - c)\,(h_1 + y_1 - y_2) = \text{constant}$$
$$(u_2 - c)\,(h_2 + y_2) \qquad = \text{constant}$$

whence, if we ignore squares and products of the small quantities u and y, we get, as above

$$u_1h_1 - c(y_1 - y_2) = \text{constant} = 0 \quad . \quad . \quad . \quad (29.4\text{h})$$

$$u_2h_2 - cy_2 \qquad = \text{constant} = 0 \quad . \quad . \quad . \quad (29.4\text{i})$$

Substituting from (29.4h) in (29.4f) so as to eliminate u_1 we get

$$gy_1 - \frac{c^2(y_1 - y_2)}{h_1} = 0$$

whence

$$y_1(gh_1 - c^2) + c^2y_2 = 0 \quad . \quad . \quad . \quad . \quad (29.4\text{j})$$

From this we see at once that if c^2 is very small then y_1 must be very small relatively to y_2. In other words, there is thus indicated the possibility of large oscillations in the surface of separation, unaccompanied by large oscillations in the free surface; or again, there may be large internal waves without large oscillations of the free surface. (It should be noted that we are dealing throughout with long waves whose length is much greater than the depth.)

Similarly, we find, from (29.4i) and (29.4g)

$$\frac{\rho_1}{\rho_2}y_1 + y_2\left(1 - \frac{\rho_1}{\rho_2} - \frac{c^2}{gh_2}\right) = 0 \quad . \quad . \quad . \quad (29.4\text{k})$$

This indicates also that if c is small compared with $\sqrt{gh_2}$, then y_1 is much smaller than y_2 in the ratio

$$\frac{\rho_2 - \rho_1}{\rho_1}$$

From the last two equations we can eliminate y_2 and so obtain

$$y_1\left[\frac{\rho_1}{\rho_2} - \left(\frac{gh_1 - c^2}{c^2}\right)\left(1 - \frac{\rho_1}{\rho_2} - \frac{c^2}{gh_2}\right)\right] = 0 \quad . \quad . \quad (29.4\text{l})$$

which gives us an equation for determining the rate of propagation c, for if we suppose that y_1 is not zero then the quantity in square brackets must be zero, whence we obtain, after a little reduction

$$\frac{c^2}{gh_2} + \frac{gh_1}{c^2}\left(1 - \frac{\rho_1}{\rho_2}\right) = 1 + \frac{h_1}{h_2} \quad . \quad . \quad . \quad . \quad (29.4\text{m})$$

or

$$c^4 - g(h_1 + h_2)c^2 + g^2h_1h_2\left(1 - \frac{\rho_1}{\rho_2}\right) = 0 \quad . \quad . \quad . \quad (29.4\text{n})$$

It will be noted that equation (29.4n) gives two values of c^2 for a free wave. If, as is usually the case, the difference in density between the two strata is very small, then the last term in the equation is very small, and it is a simple deduction to show that we must have, very approximately, the alternative values.

$$c^2 = g(h_1 + h_2) \quad . \quad . \quad . \quad . \quad . \quad (29.4\text{p})$$

or

$$c^2 = \frac{gh_1h_2}{h_1 + h_2}\left(1 - \frac{\rho_1}{\rho_2}\right) \quad . \quad . \quad . \quad . \quad (29.4\text{q})$$

The first possible rate of propagation of free waves is that which we should expect in a fluid of constant density, of the same total depth; that is, we get the familiar expression $c^2 = gh$, where $h = h_1 + h_2$.

The second expression gives a very slow rate of propagation.

In the first case we have, from (29.4j), the ratios of elevations given by

$$\frac{y_2}{y_1} = \frac{h_2}{h_1 + h_2} \quad . \quad . \quad . \quad . \quad . \quad (29.4\text{r})$$

so that the amplitude of the wave on the free surface is greater than that of the internal wave. In such a case there would be nothing abnormal.

Substituting for the case of slow movement from (29.4q) and (29.4k) we get

$$\frac{y_2}{y_1} = \frac{h_1 + h_2}{h_2} \cdot \frac{\rho_1}{\rho_1 - \rho_2} \qquad . \quad . \quad . \quad . \quad . \quad (29.4s)$$

The internal wave in this case is very much greater than the external wave, inversely as the difference in density and the ratio of the depth of the lower stratum to the whole depth.

With regard to the streams we see from (29.4h) and (29.4i) that in the two strata, for the case of large internal waves in which y_1 is small compared to y_2, the velocities of the streams have the ratio

$$\frac{u_1}{u_2} = -\frac{h_2}{h_1} \qquad . \quad . \quad . \quad . \quad . \quad (29.4t)$$

which, of course, also follows from the fact that if the surface is unchanged, the condition of equal rates of transport of volume requires

$$h_1 u_1 + h_2 u_2 = 0$$

It is clear, however, that the streams in the two strata may be very different in velocity.

REFERENCES

The following list of books and papers, and references to authors, does not profess to be a bibliography of the subject. The great majority of the readers of this Manual will probably not be interested in the mathematical papers which are fundamental to the theory of the subject, so that many important papers are omitted. It is regarded as sufficient to provide a general guide to the literature, principally in the English language, and to give references to memoirs which have been used directly or indirectly in the compilation of the Manual. Titles of papers and books have been abbreviated in some instances.

Elementary expositions of tides

DARWIN, G. H. 1911. The Tides and Kindred Phenomena, 3rd Edition. Murray, London.

MARMER, H. A. 1926. The Tide. Appleton & Co., New York and London.

The former is still an important book on the subject of the tides, though it is incomplete as regards modern methods and theories. It gives an account of " kindred phenomena " in connection with astronomical theories.

The book by Marmer is probably the best book extant in popular language for the general reader, and it is noteworthy in general for accuracy as well as lucidity.

General text-books

AIRY, G. B. 1845. Tides and Waves. *Encyclopædia Metropolitana.*

HARRIS, R. A. 1897–1907. Manual of Tides. Appendices to Reports of the U.S. Coast and Geodetic Survey.

LAMB, H. 1932. Hydrodynamics, 6th Edition. Cambridge University Press.

Airy's work is now, perhaps, principally of historic value, though he treats of many subjects not often discussed in such detail. Most of his results are reproduced by Harris.

Harris's Manual is a storehouse of information and includes many valuable tables. Unfortunately, it does not appear to have been published separately and so it is not easily accessible.

Lamb's text-book on Hydrodynamics is a standard work, probably the best available on the general subject. One chapter is devoted to tides, but other parts of the text-book refer to allied matters.

Summaries of modern theories

PROUDMAN, J. 1929. Tides. *Encyclopædia Britannica*, 14th Edition.

DOODSON, A. T. 1931. Figure of the Earth, Chapter II. National Research Council. Nat. Acad. of Sci., Washington, D.C. *Bulletin No.* 78.

THORADE, H. 1931. Probleme der Wasserwellen. *Probleme der Kosmischen Physik.* Henri Grand, Hamburg.

PROUDMAN, J., and GRACE, S. F. 1930. Historical review of dynamical explanations of tides and seiches in narrow seas and lakes. Sect. of Oceanography, Int. Union Geodesy and Geophysics. *Bulletin No.* 15.

GRACE, S. F. 1931. Historical reviews. Association of Physical Oceanography, International Union for Geodesy and Geophysics, *Sci. Pub. No.* **1**, 3–26.

HOPFNER, F. 1931. Die Gezeiten der Meere. Handbuch der Experimental-Physik 25, Geophysik 2, 689–801. Leipzig.

Bibliographies and lists

PROUDMAN, J. 1929–40. Bibliography on Tides and certain Kindred Matters. Reports published by the International Union for Geodesy and Geophysics.

INTERNATIONAL HYDROGRAPHIC BUREAU, Monaco.

(*a*) 1933 List of Harmonic Constants. *Special Publication No.* 26.

(*b*) 1935 List of tidal authorities, gauges and records. *Special Publication No.* 31.

Many of the books and memoirs referred to in this appendix contain short bibliographies, but the one prepared by Proudman is a very full one. It has appeared as *Bulletins Nos.* 12 and 17, and *Scientific Publications Nos.* 2, 3, 6 of the Association of Physical Oceanography, one of the sections of the Union. The publications of the International Hydrographic Bureau are very useful in connection with tide gauges, records, and the derived harmonic constants. The latter are given in full, whereas the Admiralty Tide Tables, Part II, only give constants for M_2, S_2, K_1, O_1, for a selected number of places, with inferred constants for many other places for which analyses have not been made.

Tide-generating forces, equilibrium tide, harmonic analysis and prediction

NEWTON, I. 1687. Philosophiæ Naturalis Principia Mathematica. (See also Proudman, J., 1927. Newton's work on the theory of the tides, in "*Isaac Newton, 1642–1727,*" Mathematical Association.)

THOMSON, W. 1868–76. Reports of Committee for harmonic analysis. Brit. Ass. for Adv. of Sci.

ROBERTS, E. 1868–76. *Ibid.*

DARWIN, G. H. 1883–86. Reports of a Committee for the harmonic analysis. Brit. Ass. for Adv. of Sci. (See also Scientific Papers, Vol. **I,** Cambridge University Press).

DOODSON, A. T. 1921. The harmonic development of the tide-generating potential. *Proc. Roy. Soc.* A **100,** pp. 305–329.

SCHUREMAN, P. 1924. Manual of harmonic analysis and prediction of tides. U.S. Coast and Geodetic Survey, *Special Publication No.* 98.

DOODSON, A. T. 1927. The analysis of tidal observations. *Phil. Trans. Roy. Soc.*, A **227,** pp. 223–279.

DOODSON, A. T. 1928. The analysis and prediction of tidal currents from observations of times of slack water. *Proc. Roy. Soc.* A **121,** pp. 72–88.

SUTHONS, C. T. 1959. The Admiralty Semi-Graphic Method of Harmonic Tidal Analysis. Admiralty Tidal Handbook No. 1

The paper by Doodson, 1921, is a very full development of the tide-generating forces, and his paper of 1927 on the analysis of tidal observations gives the principles of the methods of analysis used by the Tidal Institute and by the Admiralty. Schureman's Manual expounds the development of the potential as given by Darwin, and the methods of analysis standard in the United States ; also many useful tables are included. The 1928 paper by Doodson gives methods of analysis which have since been applied to many special problems.

Tide gauges, current meters, tide-predicting machines

THOMSON, W. 1881. The tide gauge, tidal harmonic analyser and tide predicter. *Proc. Inst. Civil Engrs.*, **65** (3).

INTERNATIONAL HYDROGRAPHIC BUREAU. 1926. Tide predicting machines. *Special Publication No.* 13, Monaco.

THORADE, H. 1933. Methoden zum Studium der Meeres-strömungen. *Handb. biol. Arbeitsmethoden* **2** (3), 2865–3095, Berlin.

The literature concerning tide gauges and current meters is very great. Tide gauges of normal type are made by many firms, and electrically operated tide gauges have been designed by Doodson in this country and Rauschelbach in Germany.

Current meters in great variety have been designed by Pillsbury, Ekman, Petterson. Jacobson, Witting, Ott, Rauschelbach, Idrac, Wollaston, Carruthers, and Doodson. The review by Thorade is a very useful summary of the salient features of most of these meters. Many articles on the subject are to be found in the volumes of the Hydrographic Review, Monaco.

Particulars of tide-predicting machines (photographs, list of components, etc.) are given in the Monaco special publication.

Tidal oscillations in channels, inlets and small seas

BERNOULLI, D. 1738. Hydrodynamica.

RAYLEIGH, LORD. 1876. On waves. *Phil. Mag.*, **1**, 257–279. (Sci. Papers, **1**, 251–271).

PROUDMAN, J. 1925. *Phil. Mag.*, **49**, 465–475.

PROUDMAN, J. 1925–28. *Mon. Not. Roy. Astr. Soc. Geophys. Supp.*, **1**, 247–270, 360–371; **2**, 22–43, 96–97, 111–119.

DEFANT, A. 1925. Gezeitenprobleme des Meeres in Landnahe. *Probleme der Kosmischen Physik.* 1–80, Hamburg.

In this Manual considerable use is made of Bernoulli's equation, first given in 1738. It is discussed in detail in all standard works on hydrodynamics or hydraulics, and finds many applications in connection with river engineering. Much use is also made of the artifice due to Rayleigh, whereby progressive waves are reduced to steady motions so that Bernoulli's equation can be used. The above two papers are the original sources. The papers by Proudman on tides in channels and small seas have been freely used.

The methods used in Chapters XVII–XIX have been specially devised for this Manual, but some mathematical investigations will be found in the general text-books.

Mathematical investigation of tides in oceans

LAPLACE, P. S. 1775–76. Recherches sur quelques points du système du monde. *Mém. de l'Acad. Roy. des Sci.* (Mécanique Celeste, edition 1799).

THOMSON, W. 1875. On an alleged error in Laplace's theory of tides. *Phil. Mag.* (4) **50**, 227–342.

GOLDSBROUGH, G. R. 1913–14. Tides in polar and zonal oceans. *Proc. Lon. Math. Soc.*, **14**, 31–66, 207–229.

GOLDSBROUGH, G. R. 1927–33. Tides in oceans on rotating globe, Parts I–IV (Part III with D. C. Colborne). *Proc. Roy. Soc. A*, **117**, 692–718; **122**, 228–245; **126**, 1–15; **140**, 241–253.

COLBORNE, D. C. 1931. Diurnal tide in ocean bounded by two meridians. *Proc. Roy. Soc. A*, **131**, 38–52.

DOODSON, A. T. 1927. Application of numerical methods of integration to tidal dynamics. *Mon. Not. Roy. Astr. Soc. Geophys. Supp.*, **1**, 541–557.

PROUDMAN, J., and DOODSON, A. T. 1927. Tides in ocean bounded by two meridians on a non-rotating earth. *Mon. Not. Roy. Astr. Soc. Geophys. Supp.*, **1**, 468–483.

PROUDMAN, J. 1935. Tides in oceans bounded by meridians, Part I. *Phil. Trans. Roy. Soc. A.*, **235**, 237–289.

DOODSON, A. T. 1935–40. Tides in oceans bounded by meridians, Parts II–V. *Phil. Trans. Roy. Soc. A*, **235**, 290–333; **237**, 311–373; **238**, 477–512.

JEFFREYS, H. 1920. Tidal friction in shallow seas. *Phil. Trans. Roy. Soc. A.*, **221**, 239–264.

JEFFREYS, H. 1923. Tidal dissipation of energy. *Nature*, **112**, 622.

Laplace's problem of tides in an ocean covering the whole earth is expounded in the paper under his name, and is discussed by Thomson (1875) and Doodson (1927). The papers by Jeffreys give computations of the dissipation of energy; the 1920 paper gives computations in relation to the moon's motion, and the 1923

letter states his view that tidal friction seriously affects the tides. The remaining papers are summarised in this Manual.

Cotidal charts and explanations of tides

WHEWELL, W. 1833. Essay towards a first approximation to a map of cotidal lines. *Phil. Trans. Roy. Soc.*, 147–236.

WHEWELL, W. 1836. On the results of an extensive series of tide observations. *Phil. Trans. Roy. Soc.*, 289–342

PROUDMAN, J. 1928. Curvature of cotidal lines across a channel. *Mon. Not. Roy. Astr. Soc. Geophys. Supp.*, **2**, 111–119.

PROUDMAN, J., and DOODSON, A. T. 1924. The principal constituent of the tides of the North Sea. *Phil. Trans. Roy. Soc., A*, **224**, 185–219.

DOODSON, A. T., and CORKAN, R. H. 1931. The principal constituent of the tides in the English and Irish Channels. *Phil. Trans. Roy. Soc., A*, **231**, 29–53.

STERNECK, R. 1920–21. Die Gezeiten der Ozeane. *Sitz. Ber. Akad. Wiss. Wien. IIa*, **129**, 131–150; **130**, 363–371.

PROUDMAN, J. 1931. The principal constituent of the tides in British waters. *Conf. Empire Survey Officers, Report*, 356–367.

Surges

EKMAN, V. W. 1905. On the influence of the earth's rotation on ocean currents. *Arkiv. Math. Astron. Fysik.* **2** (11), 1–52.

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A great deal of important work has been done recently by other Japanese writers.

A very comprehensive bibliography on tides is being published in cumulative form by the International Association of Physical Oceanography as follows:—

BIBLIOGRAPHY ON TIDES, 1665–1939. 1955. Geofysisk Institutt, Bergen, Norway.

BIBLIOGRAPHY ON TIDES, 1940–1954. 1957. Geofysiska Institutet, Goteborg, Sweden.

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Printed in England under the authority of Her Majesty's Stationery Office
by the Hydrographic Supplies Establishment.

C.B.H. 93 - Wt. 45194 - K.6 - 3/61